On Maxwell's equations in exterior domains

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Abstract:
In this paper the long time asymptotic behavior of solutions of Maxwell's equations with electric conductivity in an exterior domain with mixed boundary conditions is investigated. It is shown that the solution behaves asymptotically like a free space solution provided it obeys a suitable local decay-property. As a consequence the completeness of the wave-operators is obtained under very general assumptions on the coefficients.
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1 Introduction

The subject of this paper is the long time behavior of the solutions of Maxwell's equations

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{H} - \sigma\mathbf{E}, \quad \mu\partial_t\mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.1}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0,\infty) \times \Gamma_1 \text{ and } \vec{n} \wedge \mathbf{H} = 0 \text{ on } (0,\infty) \times \Gamma_2 \tag{1.2}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{H}(0,x) = \mathbf{H}_0(x). \tag{1.3}$$

This system describes the propagation of the electromagnetic field in a domain $\Omega \subset I\!R^3$ with bounded complement. The unknown functions $\mathbf{E}, \mathbf{H}$ denote the electric and magnetic field respectively which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$. The dielectric and magnetic susceptibilities $\varepsilon, \mu \in L^\infty(I\!R^3)$ are assumed to be uniformly positive. $\sigma \in L^\infty(\Omega)$ is the nonnegative electric conductivity, which also depends on the space-variable, $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$.

In [12] it is shown that for Maxwell's system without displacement current

$$\text{curl } \mathbf{H} = \sigma\mathbf{E}, \quad \mu\partial_t\mathbf{H} = - \text{ curl } \mathbf{E},$$

the global energy of the magnetic field $\mathbf{H}$ decays exponentially, even if the set G, where the damping by the conductivity is present, is bounded. However, this does not hold for the solutions of 1.1-1.3.

Let X_0 denote the set of all initial-states $(\mathbf{E}_0, \mathbf{H}_0) \in X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$, such that the solution $(\mathbf{E}, \mathbf{H}) \in C([0,\infty), L^2(\Omega, \mathbb{C}^6))$ of the initial-boundary-value-problem 1.1-1.3 has the local decay-property

$$t^{-1} \int_0^t \int_{\{x \in \Omega : |x| \leq R\}} \varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 dx ds \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0. \tag{1.4}$$

The basic goal of this paper is to show that for each initial-state $(\mathbf{E}_0, \mathbf{H}_0) \in X_0$ there exists a unique
$(\mathbf{F}_0, \mathbf{G}_0) \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{F}_0 = \operatorname{div} \mathbf{G}_0 = 0$, such that

$$||(\mathbf{E}(t), \mathbf{H}(t)) - (\mathbf{F}(t), \mathbf{G}(t))||_{L^2(\Omega)} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{1.5}$$

Here $(\mathbf{F}, \mathbf{G}) \in C(\mathbb{R}, L^2(\mathbb{R}^3, \mathbb{C}^6))$ denotes the solution to Maxwell's equations in the whole space, that is

$$\partial_t \mathbf{F} = \operatorname{curl} \mathbf{G}, \quad \partial_t \mathbf{G} = -\operatorname{curl} \mathbf{F}, \tag{1.6}$$

supplemented by the initial-condition

$$\mathbf{F}(0,x) = \mathbf{F}_0(x), \mathbf{G}(0,x) = \mathbf{G}_0(x). \tag{1.7}$$

This means that the solution to 1.1-1.3 behaves asymptotically like a free space solution to equations 1.6, 1.7 as $t \to \infty$ if $(\mathbf{E}, \mathbf{H})$ has the local decay-property 1.4.

For this purpose it is assumed that ε and μ tend to 1 as $|x| \to \infty$, more precisely

$$(1+|x|)^{(1+\delta)}(\varepsilon - 1) \in L^{r_0}(\mathbb{R}^3), \quad (1+|x|)^{(1+\delta)}(\mu - 1) \in L^{r_0}(\mathbb{R}^3) \tag{1.8}$$

and $(1+|x|)^{(1+\delta)}\sigma \in L^{r_0}(\mathbb{R}^3)$ for some $r_0 \in [3,\infty]$ and $\delta > 0$.
This includes in particular a medium occupying an unbounded set $G \subset \Omega$ with constant susceptibilities $\varepsilon \neq 1$, $\mu \neq 1$, provided that
$\operatorname{meas}(\{x \in G : |x| \geq r\})$ decays sufficiently fast as $r \to \infty$.
The main step in the proof of 1.5 is to show that for all $(\mathbf{E}_0, \mathbf{H}_0) \in X_0$ and $\alpha < 1$ one has

$$\int_{\{x \in \Omega : |x| \leq \alpha t\}} \varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0.$$

In section 3 a L^p-regularity theorem for solutions to 1.1-1.3 is proved, which will be used in the proof of 1.5 .

The subject of section 6 are sufficient conditions for the local energy-decay property 1.4. For this purpose it is assumed that the set
$G \stackrel{\text{def}}{=} \{x \in \Omega : \sigma(x) > 0\}$ has nonempty interior and $\varepsilon(x) = \mu(x) = 1$ for all $x \in \mathbb{R}^3 \setminus G$. The local decay property

$$\int_{\{x \in \Omega : |x| \leq R\}} \varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0 \tag{1.9}$$

is shown for all initial-data $\mathbf{E}_0 \in L^2(\Omega)$, $\mathbf{H}_0 \in L^2(\Omega)$, which obey

$$\int_\Omega (\varepsilon \mathbf{E}_0 \mathbf{f} + \mu \mathbf{H}_0 \mathbf{g}) dx = 0 \text{ for all } (\mathbf{f}, \mathbf{g}) \in \mathcal{N}. \tag{1.10}$$

Here $\mathcal{N} \subset L^2(\Omega)$ denotes the set of all $(\mathbf{f}, \mathbf{g}) \in L^2(\Omega)$ with

$$\text{curl } \mathbf{f} = \text{ curl } \mathbf{g} = 0 \text{ on } \Omega, \quad \vec{n} \wedge \mathbf{f} = 0 \text{ on } \Gamma_1, \vec{n} \wedge \mathbf{g} = 0 \text{ on } \Gamma_2$$

$$\text{and } \mathbf{f} = 0 \text{ on } G.$$

Condition 1.10 includes

$$\text{div } (\varepsilon \mathbf{E}_0) = 0 \text{ on } \Omega_0 \stackrel{\text{def}}{=} \Omega \setminus G \text{ and } \text{ div } (\mu \mathbf{H}_0) = 0 \text{ on } \Omega.$$

Note that $\mathcal{N} \subset L^2(\Omega)$ is the set of stationary states for problem 1.1-1.3.

The local decay-property 1.9 follows from the result in [14] concerning decay in the weak topology together with the compactness-result in [9]. In [14] weak decay is shown using a suitable modification of the method for wave-equations in bounded domains in [5], where the ω-limit set of the trajectories is investigated, see also [2], [8].

In the last section 7 the undamped case, i.e. $\sigma = 0$ is considered. Let $\exp(tB), t \in I\!R$ be the unitary group associated with the Cauchy-problem 1.1-1.3 and $X_{cont}(iB)$ the continuous subspace for the self-adjoint operator iB, where B is the generator of $\exp(tB), t \in I\!R$. Let $J : L^2(I\!R^3) \to L^2(\Omega)$ be the restriction operator and $\exp(t\mathcal{B}_0), t \in I\!R$ be the unitary group for the free-space-problem 1.6-1.7 and $\mathcal{P}_0$ be the orthogonal-projector on $(\ker \mathcal{B}_0)^\perp$, the space of the dynamical modes $(\mathbf{F}, \mathbf{G}) \in L^2(I\!R^3)$ for 1.6, 1.7 satisfying div $\mathbf{F} =$ div $\mathbf{G} = 0$. 1.5 will be used to prove the completeness of the wave-operators, i.e.

$$\text{ran } \Omega^+ = X_{cont}(iB), \tag{1.11}$$

where

$$\Omega^+ \stackrel{\text{def}}{=} \lim_{t \to \infty} \exp(-tB) J \exp(t\mathcal{B}_0) \mathcal{P}_0.$$

(defined as the stong limit in $L^2(\Omega)$.) This directly implies that $X_{cont}(iB) = X_{ac}(iB)$, where $X_{ac}(iB)$ denotes the absolutely continuous subspace for B. 1.11 or the weaker assertion ran $\Omega^+ = X_{ac}(iB)$ are well known in the case that the coefficients ϵ and μ are smooth , see [3], [16], [20], [21], [23], where trace-class perturbation theorems are used . In [10] 1.11 is obtained in the case that the the coefficients are not necessarily smooth, but it is assumed that they are equal 1 outside some bounded set . In this paper it is shown that only assumption 1.8 without any further regularity conditions on the coefficients is sufficient to obtain directly 1.11.

2 Notation, Assumptions

For an arbitrary open set $K \subset I\!R^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$.
$H_{curl}(K)$ is defined as the space of all $\mathbf{E} \in L^2(K, \mathbb{C}^3)$ with $\operatorname{curl} \mathbf{E} \in L^2(K)$ endowed with the norm

$$||\mathbf{E}||^2_{H_{curl}(K)} \stackrel{\text{def}}{=} ||\mathbf{E}||^2_{L^2(K)} + ||\operatorname{curl} \mathbf{E}||^2_{L^2(K)}.$$

For $p \in [1, \infty)$ the dual exponent p^* is given by $p^{-1} + (p^*)^{-1} = 1$.
Let $\Omega \subset I\!R^3$ be a (connected) domain with bounded complement, such that $I\!R^3 \setminus \overline{\Omega}$ is a Lipschitz-domain and let $\varepsilon, \mu \in L^\infty(I\!R^3)$ be uniformly positive functions.

In the sequel we denote by $\underline{\mathbf{w}}_1 \in \mathbb{C}^3$ the first three and by $\underline{\mathbf{w}}_2 \in \mathbb{C}^3$ the last three components of a vector $\mathbf{w} \in \mathbb{C}^6$ and $S\mathbf{w} \stackrel{\text{def}}{=} (-x \wedge \underline{\mathbf{w}}_2, x \wedge \underline{\mathbf{w}}_1)$.

Moreover, $(\varepsilon^{-1/2}\underline{\mathbf{w}}_1, \mu^{-1/2}\underline{\mathbf{w}}_2)$ is denoted by $E\mathbf{w}$ for $\mathbf{w} \in L^2(\Omega, \mathbb{C}^6)$ or $\mathbf{w} \in L^2(I\!R^3, \mathbb{C}^6)$.

Next, some function-spaces related to Maxwell's equations with mixed boundary-conditions are introduced.

W_H denotes the closure of $C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{curl}(\Omega)$, where $H_{curl}(\Omega)$, is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with $\operatorname{curl} \mathbf{E} \in L^2(\Omega)$.
W_E denotes the set of all $\mathbf{E} \in H_{curl}(\Omega)$, such that

$$\int_\Omega \mathbf{E} \operatorname{curl} \mathbf{F} - \mathbf{F} \operatorname{curl} \mathbf{E} dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1, see [9].

Now, the following operators are defined.
Let $D(B_0) \stackrel{\text{def}}{=} W_E \times W_H$ and

$$B_0(\mathbf{E}, \mathbf{H}) \stackrel{\text{def}}{=} (\operatorname{curl} \mathbf{H}, -\operatorname{curl} \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{H}) \in D(B_0).$$

Next, define $B \stackrel{\text{def}}{=} EB_0E$ i. e.

$$B(\mathbf{F}, \mathbf{G}) \stackrel{\text{def}}{=} (\varepsilon^{-1/2} \operatorname{curl} (\mu^{-1/2}\mathbf{G}), -\mu^{-1/2} \operatorname{curl} (\varepsilon^{-1/2}\mathbf{F}))$$

for $E(\mathbf{F}, \mathbf{G}) \in D(B_0) = W_E \times W_H$. It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the usual scalar-product .

Let P be the orthogonal projector on $(\ker B)^\perp = \overline{\operatorname{ran} B}$.

By setting $\mathbf{u} \stackrel{\text{def}}{=} (\varepsilon^{1/2}\mathbf{E}, \mu^{1/2}\mathbf{H})$ problem 1.1-1.3 reads as

$$\partial_t \underline{\mathbf{u}}_1 = \varepsilon^{-1/2} \operatorname{curl} (\mu^{-1/2}\underline{\mathbf{u}}_2) - \varepsilon^{-1}\sigma\underline{\mathbf{u}}_1, \tag{2.12}$$

$$\partial_t \underline{\mathbf{u}}_2 = -\mu^{-1/2} \operatorname{curl} (\varepsilon^{-1/2}\underline{\mathbf{u}}_1)$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \underline{\mathbf{u}}_1 = 0 \text{ on } (0, \infty) \times \Gamma_1, \quad \vec{n} \wedge \underline{\mathbf{u}}_2 = 0 \text{ on } (0, \infty) \times \Gamma_2, \tag{2.13}$$

$$\mathbf{u}(0,x) = \mathbf{w}(x). \tag{2.14}$$

For $\mathbf{w} \in X = L^2(\Omega, \mathbb{C}^6)$ a function $\mathbf{u} \in C(\mathbb{R}, X)$ is called a weak soution to 2.12-2.14 if

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X + \langle F_\sigma \mathbf{u}(t), \mathbf{a}\rangle_X \text{ for all } \mathbf{a} \in D(B) \tag{2.15}$$

and 2.14 is fulfilled. Here $F_\sigma : X \to X$ is defined by

$$F_\sigma \mathbf{u} \stackrel{\text{def}}{=} -\varepsilon^{-1}\sigma \cdot (\underline{\mathbf{u}}_1, 0).$$

2.15 is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp(tB)\mathbf{w} + \int_0^t \exp((t-s)B) F_\sigma \mathbf{u}(s) ds \tag{2.16}$$

where $(\exp(tB))_{t\in\mathbb{R}}$ is the unitary group generated by B. By the assumption on σ the operator $-F_\sigma$ is bounded and positive in X in the sense that

$$\text{re } \langle F_\sigma \mathbf{u}, \mathbf{u}\rangle_X \leq 0 \text{ for all } \mathbf{u} \in X.$$

Since B is skew-self-adjoint in X, it follows that $B + F_\sigma$ is (maximal) dissipative in X, i.e. $(0, \infty) \subset \rho(B + F_\sigma)$ and $||(\lambda - B - F_\sigma)^{-1}||_{B(X)} \leq \lambda^{-1}$ for all $\lambda \geq 0$, where $\rho(B+F_\sigma)$ denotes the resolvent-set and $||\cdot||_{B(X)}$ the operator-norm in X. Hence $B + F_\sigma$ is the generator of a strongly continuous, contractive semigroup $(\exp(t(B + F_\sigma)))_{t\geq 0}$ on X, see [18] and the solution of 2.12-2.14 is given by

$$\mathbf{u}(t) = T(t)\mathbf{w} \stackrel{\text{def}}{=} \exp(t(B + F_\sigma))\mathbf{w}. \tag{2.17}$$

2.16 yields the energy estimate

$$\frac{1}{2}\frac{d}{dt}||T(t)\mathbf{w}||_X^2 = \text{ re } \langle F_\sigma(T(t)\mathbf{w}), T(t)\mathbf{w}\rangle_X \tag{2.18}$$

$$= -\int_G \varepsilon^{-1}\sigma|\underline{(T(t)\mathbf{w})}_1|^2 dx \leq 0.$$

In the sequel let $R_0 > 0$, such that $\mathbb{R}^3 \setminus \Omega \subset B_{R_0} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 : |x| < R_0\}$
and choose $\chi_0 \in C^\infty(\mathbb{R}^3)$ with supp $\chi_0 \subset \Omega$ and $\chi_0(x) = 1$ on $\mathbb{R}^3 \setminus B_{R_0}$.
Then for $\mathbf{w} \in X$ or $\mathbf{w} \in L^2(\mathbb{R}^3)$ define $\mathcal{C}_0\mathbf{w} \stackrel{\text{def}}{=} ((\nabla\chi_0) \wedge \underline{\mathbf{w}}_2, -(\nabla\chi_0) \wedge \underline{\mathbf{w}}_1)$,
For convenience $\chi_0\mathbf{w}$ and $\mathcal{C}_0\mathbf{w}$ will be regarded as elements of $L^2(\mathbb{R}^3)$ by extending them by zero outside supp χ_0 if $\mathbf{w} \in X$.

Next let $D(\mathcal{B}_0) \stackrel{\text{def}}{=} H_{curl}(\mathbb{R}^3) \times H_{curl}(\mathbb{R}^3)$ and $\mathcal{B}_0(\mathbf{e}, \mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, - \text{ curl } \mathbf{e})$.
Moreover, let $\mathcal{B} \stackrel{\text{def}}{=} E\mathcal{B}_0 E$.
Finally, let $\mathcal{P}_0$ and $\mathcal{P}$ be the orthogonal projectors on $(\ker \mathcal{B}_0)^\perp$ and $(\ker \mathcal{B})^\perp$ respectively.
Obviously the unique weak solution to the whole-space-problem 1.6, 1.7 is given by

$$(\mathbf{F}(t), \mathbf{G}(t)) = \exp(t\mathcal{B}_0)(\mathbf{F}_0, \mathbf{G}_0),$$

where $(\exp(t\mathcal{B}_0))_{t\in\mathbb{R}}$ is the unitary group generated by $\mathcal{B}_0$.

3 A L^p-regularity-theorem and a commutator-estimate

In this section a L^p-regularity-theorem for elements of $D(|B|^s)\cap \ker B$ for small $s>0$ is proved, which is similar to an regularity theorem in [13] for the two-dimensional Maxwell-operator in a bounded domain $\mathcal{G}$ with mixed boundary-conditions, in which
$(\ker B)^\perp \cap (D(|B|^s)) \subset H^s(\mathcal{G})$ for small $s>0$ is proved.

Here $|B|^s \stackrel{\text{def}}{=} \int_{I\!R} |\lambda|^s dE_\lambda$ by the spectral-theorem, where $(E_\lambda)_{\lambda\in I\!R}$ is the spectral-resolution of the self-adjoint operator iB in X. It turns out that its domain $D(|B|^s)\subset X$ coincides with the (complex) interpolation space $[X, D(B)]_s$ between X and $D(B)$ for $s\in[0,1]$, see [24].
A $H^{1/2}$-regularity result for Maxwell's equations in Lipschitz-domains but without mixed boundary-conditions can be found in [4].
Recall that $D(\mathcal{B}_0) \stackrel{\text{def}}{=} H_{curl}(I\!R^3)\times H_{curl}(I\!R^3)$
and $\mathcal{B}_0(\mathbf{e},\mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, -\text{ curl } \mathbf{e})$. Moreover, $\mathcal{B} \stackrel{\text{def}}{=} E\mathcal{B}_0 E$ and $\mathcal{P}_0$ and $\mathcal{P}$ are the orthogonal projectors on $\overline{(\text{ran } \mathcal{B}_0)} = (\ker \mathcal{B}_0)^\perp$ and $\overline{(\text{ran } \mathcal{B})} = (\ker \mathcal{B})^\perp$ respectively.
In the sequel let $\mathcal{A}\subset L^2(I\!R^3)$ the space of all $\mathbf{u}\in L^2(I\!R^3)$, such that
div $(\varepsilon^{1/2}\underline{\mathbf{u}}_1)\in L^2(I\!R^3)$ and div $(\mu^{1/2}\underline{\mathbf{u}}_2)\in L^2(I\!R^3)$ in the sense of distributions.

Theorem 1 *There exists some $s_1>0$, such that for all $s\in[0,s_1]$ one has*

$$D(|\mathcal{B}|^s)\cap\mathcal{A}\subset L^{6/(3-2s)}(I\!R^3).$$

and $$||\mathbf{u}||_{L^{6/(3-2s)}(I\!R^3)}$$

$$\leq C_1\left(||\mathbf{u}||_{D(|\mathcal{B}|^s)} + ||\ div\ (\varepsilon^{1/2}\underline{\mathbf{u}}_1)||_{L^2(I\!R^3)} + ||\ div\ (\mu^{1/2}\underline{\mathbf{u}}_2)||_{L^2(I\!R^3)}\right).$$

for all $\mathbf{u}\in D(|\mathcal{B}|^s)\cap\mathcal{A}$ with some $C_1\in(0,\infty)$ independent of $\mathbf{u}$.

Proof:
Since $(\nabla\varphi,\nabla\psi)\in\ker\mathcal{B}_0$ for all $\varphi,\psi\in C_0^\infty(I\!R^3)$, it follows div $\underline{\mathbf{w}}_j=0$ for all $\mathbf{w}\in(\ker\mathcal{B}_0)^\perp$. Hence $\mathbf{w}\in H^1(I\!R^3)$ for all $\mathbf{w}\in D(\mathcal{B}_0)\cap$ $(\ker\mathcal{B}_0)^\perp$, in particular $\mathcal{P}_0E\mathbf{u}\in H^1(I\!R^3)$ for all $\mathbf{u}\in D(\mathcal{B})$, since $\mathcal{P}_0E(D(\mathcal{B}))\subset D(\mathcal{B}_0)\cap$ $(\ker\mathcal{B}_0)^\perp$. From interpolation it follows

$$\mathcal{P}_0E\mathbf{F}\in H^s(I\!R^3)\subset L^{6/(3-2s)}(I\!R^3) \text{ for all } \mathbf{F}\in D(|\mathcal{B}|^s) \text{ and } s\in[0,1]. \tag{3.19}$$

Suppose $\mathbf{f}\in\mathcal{A}$. Let $\mathbf{g}\stackrel{\text{def}}{=}(1-\mathcal{P}_0)E^{-1}\mathbf{f}\in\ker\mathcal{B}_0$.
Since $(\nabla\varphi,\nabla\psi)\in\ker\mathcal{B}_0$ for all $\varphi,\psi\in C_0^\infty(I\!R^3)$, it follows

$$\int_{I\!R^3}\left(\underline{\mathbf{g}}_1\nabla\varphi+\underline{\mathbf{g}}_2\nabla\psi\right)dx = \langle(\nabla\varphi,\nabla\psi),(1-\mathcal{P}_0)E^{-1}\mathbf{f}\rangle_{L^2(I\!R^3)}$$

$$=\langle(\nabla\varphi,\nabla\psi),E^{-1}\mathbf{f}\rangle_{L^2(I\!R^3)} = \int_{I\!R^3}\left(\varepsilon^{1/2}\underline{\mathbf{f}}_1\nabla\varphi+\mu^{1/2}\underline{\mathbf{f}}_2\nabla\psi\right)dx$$

$$= -\int_{\mathbb{R}^3} \left(\varphi \text{ div } (\varepsilon^{1/2}\underline{\mathbf{f}}_1) + \psi \text{ div } (\mu^{1/2}\underline{\mathbf{f}}_2) \right) dx$$

i.e. div $\underline{\mathbf{g}}_j \in L^2(\mathbb{R}^3)$. Since also $\mathbf{g} \in \ker \mathcal{B}_0$, i.e. curl $\underline{\mathbf{g}}_j = 0$, it follows

$$(1 - \mathcal{P}_0)E^{-1}\mathbf{f} \in H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3) \text{ for all } \mathbf{f} \in \mathcal{A} \text{ and } p \in (2, 6). \tag{3.20}$$

Next, it is shown that $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ is an invariant subspace under $\mathcal{P}_0$. By using Fourier-transform it follows that $\mathbf{F} \in L^2(\mathbb{R}^3)$ obeys $\mathbf{F} \in \ker \mathcal{B}_0$, if and only if $k \wedge \widehat{\underline{\mathbf{F}}}_j(k) = 0$, i.e. $\widehat{\underline{\mathbf{F}}}_j(k) = |k|^{-2}\left(k \cdot \widehat{\underline{\mathbf{F}}}_j(k)\right) k$. Hence the orthogonal-projector $1 - \mathcal{P}_0$ on $\ker \mathcal{B}_0$ is given by

$$(1 - \mathcal{P}_0)\mathbf{F} = \mathcal{F}^{-1}\left(|k|^{-2}\left([k \cdot \widehat{\underline{\mathbf{F}}}_1(k)]k, [k \cdot \widehat{\underline{\mathbf{F}}}_2(k)]k\right)\right) = \mathcal{F}^{-1}(g\hat{\mathbf{F}}), \tag{3.21}$$

where $g \in C^\infty(\mathbb{R}^3 \setminus \{0\})$, such that $|x|^{|\alpha|}\partial^\alpha g$ is bounded for all $\alpha \in \mathbb{N}_0^3$. By the multiplier-theorem, [17], [24] g is a L^p-Fourier-multiplier and it follows from 3.21 that

$$\mathcal{P}_0\mathbf{F} \in L^p(\mathbb{R}^3) \text{ and } ||\mathcal{P}_0\mathbf{F}||_{L^p(\mathbb{R}^3)} \leq C^{p-2}||\mathbf{F}||_{L^p(\mathbb{R}^3)} \tag{3.22}$$

for all $\mathbf{F} \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ and $p \in [2, 3]$ with some constant C independent of $\mathbf{F}$ and p. The latter inequality can be obtained by interpolation using $||\mathcal{P}_0||_{B(L^2,L^2)} \leq 1$.
Let $m \stackrel{\text{def}}{=} ||E||^2_{L^\infty}$ and $\mu > 0$ with $|E(x)\mathbf{y}|^2 \geq \mu|\mathbf{y}|^2$. Then

$$L \stackrel{\text{def}}{=} ||1 - \mu m^{-2}E^2||_{L^\infty} < 1 \tag{3.23}$$

By 3.22 and 3.23 it follows that

$$||\mathcal{P}_0||_{B(L^p,L^p)}||1 - \mu m^{-2}E^2||_{L^\infty} \stackrel{\mathrm{p}\to 2}{\longrightarrow} L < 1$$

Now, choose $\bar{p} > 2$ with

$$||\mathcal{P}_0||_{B(L^p,L^p)}||1 - \mu m^{-2}E^2||_{L^\infty} \leq (L+1)/2 < 1 \text{ for all } p \in [0, \bar{p}] \tag{3.24}$$

In the sequel let $\bar{s} \stackrel{\text{def}}{=} 3/2 - 3/\bar{p}$.
Suppose now $s \in [0, \bar{s}]$, $\mathbf{F} \in D(|\mathcal{B}|^s) \cap \mathcal{A}$ and set $p \stackrel{\text{def}}{=} 6/(3 - 2s) \leq \bar{p}$. Then 3.20 yields

$$(1 - \mathcal{P}_0)E^{-1}\mathbf{F} \in H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3). \tag{3.25}$$

Hence it suffices to show that

$$\mathbf{u}_1 \stackrel{\text{def}}{=} \mathcal{P}_0E^{-1}\mathbf{F} \in L^p(\mathbb{R}^3). \tag{3.26}$$

By 3.19, 3.22 and 3.25 one has

$$\mathbf{f} \stackrel{\text{def}}{=} \mathcal{P}_0E^2\mathbf{u}_1 = \mathcal{P}_0E\mathbf{F} - \mathcal{P}_0E^2(1 - \mathcal{P}_0)E^{-1}\mathbf{F} \in L^p(\mathbb{R}^3). \tag{3.27}$$

In order to use a fixed-point-argument the space $Y^p \stackrel{\text{def}}{=} (\ker \mathcal{B}_0)^\perp \cap L^p(\mathbb{R}^3)$ endowed with the norm

$$||\mathbf{u}||_{Y^p} \stackrel{\text{def}}{=} 2^{-1}\left(||\mathbf{u}||_{L^p(\mathbb{R}^3)} + ||\mathbf{u}||_{L^2(\mathbb{R}^3)}\right).$$

is introduced. Let $Q : Y^p \to (\ker \mathcal{B}_0)^\perp$ by

$$Q\mathbf{u} \stackrel{\text{def}}{=} \mathcal{P}_0 \left(\left[1 - \mu m^{-2} E^2 \right] \mathbf{u} + \mu m^{-2} \mathbf{f} \right) = \mathbf{u} - \mu m^{-2} \mathcal{P}_0 E^2 (\mathbf{u} - \mathbf{u}_1).$$

Then 3.22 and 3.27 yield

$$Q\mathbf{u} \in L^p(I\!R^3)$$

and by 3.24 one has

$$||Q\mathbf{u} - Q\mathbf{v}||_{Y^p} \le 2^{-1} ||1 - \mu m^{-2} E^2||_{L^\infty} \left(||\mathcal{P}_0||_{B(L^p, L^p)} ||\mathbf{u} - \mathbf{v}||_{L^p(I\!R^3)} + ||\mathbf{u} - \mathbf{v}||_{L^2(I\!R^3)} \right)$$

$$\le (L+1)/2 ||\mathbf{u} - \mathbf{v}||_{Y^p}.$$

Since $(L+1)/2 < 1$ there exists a unique $\mathbf{u}_0 \in Y^p$ with
$\mathbf{u}_0 = Q\mathbf{u}_0 = \mathbf{u}_0 - \mu m^{-2} \mathcal{P}_0 E^2 (\mathbf{u}_0 - \mathbf{u}_1)$, and hence

$$\mathcal{P}_0 E^2 (\mathbf{u}_0 - \mathbf{u}_1) = 0 \tag{3.28}$$

Since $\mathbf{u}_0 - \mathbf{u}_1 \in (\ker \mathcal{B}_0)^\perp$, one obtains from 3.28

$$0 = \langle \mathcal{P}_0 E^2 [\mathbf{u}_0 - \mathbf{u}_1], \mathbf{u}_0 - \mathbf{u}_1 \rangle_{L^2(I\!R^3)} = ||E[\mathbf{u}_0 + \mathbf{u}_1]||^2_{L^2(I\!R^3)},$$

in particular $\mathbf{u}_1 = \mathbf{u}_0 \in Y^p \subset L^p(I\!R^3)$, whence 3.26.

□

Theorem 2 *$D(|B|^s) \cap (ker\ B)^\perp \subset L^{6/(3-2s)}(I\!R^3 \setminus B_{R_0})$ for all $s \in [0, \bar{s}]$.
where $\bar{s} > 0$ as in the previous theorems.*

Proof:
Recall that $\chi_0 \in C^\infty(I\!R^3)$ with supp $\chi_0 \subset \Omega$ and $\chi_0(x) = 1$ for $|x| \ge R_0$.
Suppose $\mathbf{f} \in D(|B|^s) \cap (\ker\ B)^\perp$.
Since $E^{-1}(\nabla\varphi, \nabla\psi) \in \ker\ B$ for all $\varphi, \psi \in C_0^\infty(\Omega)$ and $\mathbf{f} \in (\ker\ B)^\perp$ it follows div $(\varepsilon^{1/2} \underline{\mathbf{f}}_1) =$ div $(\mu^{1/2} \underline{\mathbf{f}}_2) = 0$. Hence

$$\chi_0 \mathbf{f} \in \mathcal{A} \tag{3.29}$$

Since $\chi_0 D(B) \subset D(\mathcal{B})$, it follows by interpolation that $\chi_0 \mathbf{f} \in D(|\mathcal{B}|^s)$. Finally, Theorem 1 and 3.29 yield $\chi_0 \mathbf{f} \in L^{6/(3-2s)}(I\!R^3)$, which completes the proof.

□

Recall that $\mathcal{P}_0$ is the orthogonal-projector on $(\ker \mathcal{B}_0)^\perp$ and that $\chi_0 \in C^\infty(I\!R^3)$ with $\chi_0(x) = 1$ on $I\!R^3 \setminus B_{R_0}$ and supp $\chi_0 \subset \Omega$.

Let

$$A_\delta \mathbf{f} \stackrel{\text{def}}{=} (1 + |x|)^{1+\delta} \mathcal{P}_0 \mathbf{f} - \mathcal{P}_0((1 + |x|)^{1+\delta} \mathbf{f}) \text{ for } \mathbf{f} \in C_0^\infty(I\!R^3) \text{ and } \delta \in (0, \infty).$$

Lemma 1 *Let $\delta \geq 0$.*
Then there exists a constant $C_\delta \in (0,\infty)$ with

$$|(A_\delta \mathbf{f})(x)| \leq C_\delta \int_{I\!R^3} |x-y|^{-2}(1+|y|)^\delta |\mathbf{f}(y)|dy + C_\delta (1+|x|)^\delta \int_{I\!R^3} |x-y|^{-2}|\mathbf{f}(y)|dy$$

for all $\mathbf{f} \in C_0^\infty(I\!R^3)$.

Proof:
Since $(\nabla\varphi, \nabla\psi) \in \ker\ \mathcal{B}_0$ for all $\varphi, \psi \in C_0^\infty(I\!R^3)$, it follows $\text{ div } \underline{\mathbf{g}}_j = 0$ for all $\mathbf{g} \in (\ker\ \mathcal{B}_0)^\perp$ and therefore

$$\Delta \mathcal{P}_0 \mathbf{f} = \mathcal{B}_0^2 \mathcal{P}_0 \mathbf{f} = \mathcal{B}_0^2 \mathbf{f} = -\left(\text{ curl } \text{ curl } \underline{\mathbf{f}}_1, \text{ curl } \text{ curl } \underline{\mathbf{f}}_2 \right).$$

Hence the integral-representation

$$4\pi(\mathcal{P}_0 \mathbf{f})(x) = \int_{I\!R^3} |x-y|^{-3} \left((x-y) \wedge (\text{ curl } \underline{\mathbf{f}}_1)(y), (x-y) \wedge (\text{ curl } \underline{\mathbf{f}}_2)(y) \right) dy \tag{3.30}$$

holds. Hence

$$4\pi |\underline{(A_\delta \mathbf{f})(x)}_1| = |(1+|x|)^{1+\delta} \int_{I\!R^3} |x-y|^{-3}(x-y) \wedge \text{ curl } \underline{\mathbf{f}}_1(y) dy$$
$$- \int_{I\!R^3} |x-y|^{-3}(x-y) \wedge \text{ curl } \left((1+|y|)^{1+\delta} \underline{\mathbf{f}}_1(y) \right) dy|$$
$$\leq C_{1,\delta} \int_{I\!R^3} |x-y|^{-2}(1+|y|)^\delta |\mathbf{f}(y)| dy$$
$$+ C_{1,\delta} | \int_{I\!R^3} \left((1+|x|)^{1+\delta} - (1+|y|)^{1+\delta} \right) |x-y|^{-3}(x-y) \wedge \text{ curl } \underline{\mathbf{f}}_1(y) dy|$$

With the estimate

$$|(1+|x|)^{1+\delta} - (1+|y|)^{1+\delta}| \leq C_{2,\delta}|x-y| \left((1+|y|)^\delta + (1+|x|)^\delta \right)$$

it follows

$$|\nabla_y \left[\left((1+|x|)^{1+\delta} - (1+|y|)^{1+\delta} \right) |x-y|^{-3}(x-y) \right] |$$
$$\leq C_{3,\delta}|x-y|^{-2} \left((1+|y|)^\delta + (1+|x|)^\delta \right)$$

in particular

$$\left((1+|x|)^{1+\delta} - (1+|\cdot|)^{1+\delta} \right) |x-\cdot|^{-3}(x-\cdot) \in W^{1,1}_{loc}(I\!R^3) \text{ for fixed } x \in I\!R^3$$

Hence one obtains after partial integration

$$|\underline{(A_\delta \mathbf{f})(x)}_1|$$
$$\leq C_\delta \int_{I\!R^3} |x-y|^{-2} \left((1+x^2)^{\delta/2} + (1+y^2)^{\delta/2} \right) |\mathbf{f}(y)| dy.$$

A similar estimate for $\underline{(A_\delta \mathbf{f})(x)}_2$ completes the proof.

$\square$

Theorem 3 *Let $q \in (1,3)$, $0 < \delta < \delta_1$ and $p \stackrel{\text{def}}{=} (1/q - 1/3)^{-1}$. Then there exists a constant $C_{p,\delta,\delta_1} \in (0,\infty)$ with*

$$||A_\delta \mathbf{f}||_{L^p(I\!R^3)} \le C_{p,\delta,\delta_1} ||(1+|x|)^{\delta_1}\mathbf{f}||_{L^q(I\!R^3)} \text{ for all } \mathbf{f} \in C_0^\infty(I\!R^3).$$

in particular

$$||A_\delta \mathbf{f}||_{L^2(I\!R^3)} \le C_{2,\delta,\delta_1} ||(1+|x|)^{\delta_1}\mathbf{f}||_{L^{6/5}(I\!R^3)} \text{ for all } \mathbf{f} \in C_0^\infty(I\!R^3).$$

Proof:

For all $\mathbf{f} \in C_0^\infty(I\!R^3)$ one has by the previous lemma

$$|(A_\delta \mathbf{f})(x)| \le C_\delta (f(x) + g(x)) \tag{3.31}$$

with

$$f(x) \stackrel{\text{def}}{=} \int_{I\!R^3} |x-y|^{-2}(1+|y|)^\delta |\mathbf{f}(y)| dy$$
$$+(1+|x|)^\delta \left(\int_{B_1(x)} |x-y|^{-2}|\mathbf{f}(y)|dy + \int_{I\!R^3 \setminus B_{|x|/2}(0)} |x-y|^{-2}|\mathbf{f}(y)|dy \right)$$

and

$$g(x) \stackrel{\text{def}}{=} (1+|x|)^\delta \int_{B_{|x|/2}(0)\setminus B_1(x)} |x-y|^{-2}|\mathbf{f}(y)|dy.$$

Now

$$f(x) \le \int_{I\!R^3} |x-y|^{-2}(1+|y|)^\delta |\mathbf{f}(y)|dy$$
$$+ \int_{B_1(x)} (2+|y|)^\delta |x-y|^{-2}|\mathbf{f}(y)|dy$$
$$+ \int_{I\!R^3 \setminus B_{|x|/2}(0)} (1+|2y|)^\delta |x-y|^{-2}|\mathbf{f}(y)|dy$$
$$\le C_1 \int_{I\!R^3} |x-y|^{-2}(1+|y|)^\delta |\mathbf{f}(y)|dy$$

Since $|x|^{-2} \in L_w^{3/2}(I\!R^3)$ and $1/q + 2/3 = 1 + 1/p$, it follows

$$||f||_{L^p(I\!R^3)} \le C_1 || \, |\cdot|^{-2} * \left[(1+|\cdot|)^\delta |\mathbf{f}|\right] ||_{L^p(I\!R^3)} \le C_2 ||(1+|x|)^\delta \mathbf{f}||_{L^q(I\!R^3)} \tag{3.32}$$

by the generalized Young-inequality. Next,

$$g(x) \le (1+|x|)^\delta \min\{1, (|x|/2)^{-2}\} \int_{B_{|x|/2}(0)} |\mathbf{f}|dy$$
$$\le C_3 (1+|x|)^{\delta-2} ||(1+|y|)^{-\delta_1}||_{L^{q^*}(B_{|x|/2}(0))} ||(1+|y|)^{\delta_1}\mathbf{f}||_{L^q(B_{|x|/2}(0))}$$
$$\le C_4 (1+|x|)^{\delta-\delta_1-2+3/q^*} ||(1+|y|)^{\delta_1}\mathbf{f}||_{L^q(I\!R^3)}.$$

Since $\delta - \delta_1 - 2 + 3/q^* = \delta - \delta_1 - 3/p < -3/p$, one has $(1+|x|)^{\delta-\delta_1-2+3/q^*} \in L^p(I\!R^3)$ and hence

$$||g||_{L^p(I\!R^3)} \le C_5 ||(1+|x|)^{\delta_1}\mathbf{f}||_{L^q(I\!R^3)} \tag{3.33}$$

Finally, the estimate follows from 3.31-3.33.

□

4 Asymptotic behavior of solutions

In the sequel the following assumptions are imposed on E and σ, (which are not necessary for the previous sections).

$$(1+|x|)^{(1+\alpha_0)}(E-1) \in L^{r_0}(\mathbb{R}^3) \text{ and } (1+|x|)^{(1+\alpha_0)}\sigma \in L^{r_0}(\mathbb{R}^3) \tag{4.34}$$

for some $r_0 \in [3,\infty)$ and $\alpha_0 > 0$.

Let X_0 be as in the introduction the space of all $\mathbf{w} \in X$ with

$$t^{-1}\int_0^t ||T(s)\mathbf{w}||^2_{L^2(\Omega\cap B_R)}ds \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0.$$

For convenience also the space $X_{0,1}$ is introduced as the set of all $\mathbf{w} \in X$ with

$$||T(t)\mathbf{w}||_{L^2(\Omega\cap B_R)} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all } R > 0.$$

Later it will be shown that under the present assumtions the spaces $X_{0,1}$ and X_0 actually coincide. Moreover, let $\mathcal{N} \subset X = L^2(\Omega, \mathbb{C}^6)$ denote the set of all $(\mathbf{E},\mathbf{F}) \in \ker B$ with $\mathbf{E} = 0$ on $G \overset{\text{def}}{=} \{x \in \Omega : \sigma(x) > 0\}$.

The main goal of this section is to show that for all $\mathbf{w} \in X_0$ and $a \in (0,1)$ one has

$$\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx \overset{t\to\infty}{\longrightarrow} 0. \tag{4.35}$$

The physical meaning of 4.35 is that the wave-packet $T(t)\mathbf{w}$ is concentated near the sphere $|x| = t$ as $t \to \infty$.

In the sequel let $\alpha \overset{\text{def}}{=} \alpha_0/2$ and $q_0 \overset{\text{def}}{=} \left(r_0^{-1} + 2^{-1}\right)^{-1}$ with $\alpha_0 > 0$ and $r_0 \geq 3$ as in assumption 4.34. The following estimate will be used frequently.

Lemma 2 *There exists a constant $K_1 \in (0,\infty)$ such that*

$$\left|\int_{\mathbb{R}^3}(1+|x|)^{1+\alpha}\mathbf{f}\cdot\mathcal{P}_0\mathbf{g}dx\right| \leq K_1||\mathbf{f}||_{H^1(\mathbb{R}^3)}||(1+|x|)^{1+\alpha_0}\mathbf{g}||_{L^{q_0}(\mathbb{R}^3)}$$

for all $\mathbf{f},\mathbf{g} \in C_0^\infty(\mathbb{R}^3)$.

Proof:
Choose $\alpha_1 \in (\alpha,\alpha_0)$. Since $q_0 \geq 6/5$ one has $q_0^* \leq 6$. Therefore, it follows from Hölder's inequality, Theorem 3 and the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{q_0^*}(\mathbb{R}^3)$ that

$$\left|\int_{\mathbb{R}^3}(1+|x|)^{1+\alpha}\mathbf{f}\cdot\mathcal{P}_0\mathbf{g}dx\right| \leq \left|\langle\mathcal{P}_0\mathbf{f},(1+|x|)^{1+\alpha}\mathbf{g}\rangle_{L^2(\mathbb{R}^3)} + \langle\mathbf{f},A_\alpha\mathbf{g}\rangle_{L^2(\mathbb{R}^3)}\right| \tag{4.36}$$

$$\leq C_1||\mathcal{P}_0\mathbf{f}||_{H^1(\mathbb{R}^3)}||(1+|x|)^{1+\alpha}\mathbf{g}||_{L^{q_0}(\mathbb{R}^3)} + ||\mathbf{f}||_{L^2(\mathbb{R}^3)}||A_\alpha\mathbf{g}||_{L^2(\mathbb{R}^3)}$$

$$\leq C_1||\mathbf{f}||_{H^1(\mathbb{R}^3)}||(1+|x|)^{1+\alpha_0}\mathbf{g}||_{L^{q_0}(\mathbb{R}^3)} + C_2||\mathbf{f}||_{L^2(\mathbb{R}^3)}||(1+|x|)^{\alpha_1}\mathbf{g}||_{L^{6/5}(\mathbb{R}^3)}$$

Since $r_0 \geq 3$ one has $q_0 \geq 6/5$. Let $p \stackrel{\text{def}}{=} (5/6 - q_0^{-1})^{-1}$. Then $p \geq 3$, since $q_0 \leq 2$, and hence $(1+|x|)^{\alpha_1-\alpha_0-1} \in L^p(I\!R^3)$. With $p^{-1} + q_0^{-1} = 5/6$ this yields by Hölder's inequality

$$||(1+|x|)^{\alpha_1}\mathbf{g}||_{L^{6/5}(I\!R^3)} \leq ||(1+|x|)^{\alpha_1-\alpha_0-1}||_{L^p(I\!R^3)}||(1+|x|)^{1+\alpha_0}\mathbf{g}||_{L^{q_0}(I\!R^3)}. \tag{4.37}$$

Finally the assertion follows Lemma 4.36 and 4.37.

□

The next lemma concerns some elementary properties of $T(t) = \exp(t(B + F_\sigma))$.

Lemma 3 *i) For all* $\mathbf{w} \in X$ *one has*

$$T(\cdot)\mathbf{w} \in L^\infty((0,\infty), X) \text{ and } F_\sigma T(\cdot)\mathbf{w} \in L^2((0,\infty), X) \tag{4.38}$$

and

$$||T(t)\mathbf{w}||_X^2 + c_0||F_\sigma T(\cdot)\mathbf{w}||^2_{L^2((0,\infty),X)} \leq ||\mathbf{w}||_X^2$$

with some $c_0 \in (0,\infty)$ *independent of* $\mathbf{w}$.
ii) Suppose $\mathbf{w} \in D(B)$. *Then*

$$T(\cdot)\mathbf{w} \in W^{1,\infty}((0,\infty), X) \cap L^\infty((0,\infty), D(B)) \tag{4.39}$$

and the estimates

$$\text{and } ||\partial_t T(t)\mathbf{w}||_X + ||BT(t)\mathbf{w}||_X \leq K(||\mathbf{w}||_X + ||B\mathbf{w}||_X)$$

hold with some $K \in (0,\infty)$ *independent of* $\mathbf{w}$.
iii) $||F_\sigma T(t)\mathbf{w}||_X \stackrel{t\to\infty}{\longrightarrow} 0$ *for all* $\mathbf{w} \in X$.

Proof: Let $\mathbf{u}(t) = (\mathbf{E}(t), \mathbf{h}(t)) \stackrel{\text{def}}{=} T(t)\mathbf{w}$. Since $\sigma \in L^\infty(\Omega)$ and $\sigma \geq 0$ one has

$$\text{re } \langle F_\sigma \mathbf{f}, \mathbf{f}\rangle_X \geq c_0||F_\sigma\mathbf{f}||^2_{L^2(\Omega)} \text{ for all } \mathbf{f} \in X$$

with some $c_0 > 0$ independent of $\mathbf{f}$. Therefore, the energy estimate 2.18 yields $\frac{1}{2}\frac{d}{dt}||T(t)\mathbf{w}||_X^2 \leq -c_0||F_\sigma T(t)\mathbf{w}||_X^2$. This implies 4.38 by Gronwall's lemma.
Part ii) follows from $D(B+F_\sigma) = D(B)$ and the fact that
$T(t)\mathbf{w} = \exp(t(B+F_\sigma))\mathbf{w} \in D(B+F_\sigma)$ and
$(B+F_\sigma)T(t)\mathbf{w} = \exp(t(B+F_\sigma))(B+F_\sigma)\mathbf{w} = T(t)(B+F_\sigma)\mathbf{w}$ for $\mathbf{w} \in D(B)$.
In order to prove iii), it suffices to consider $\mathbf{w} \in D(B)$, since $D(B)$ is dense in X. Then i) and ii) yield
$F_\sigma T(\cdot)\mathbf{w} \in W^{1,\infty}((0,\infty), X) \cap L^2((0,\infty), X)$,
which implies $||F_\sigma T(t)\mathbf{w}||_X \stackrel{t\to\infty}{\longrightarrow} 0$.

□

Lemma 4 *i) $D(B) \cap X_{0,1}$ is dense in $X_{0,1}$ and $D(B) \cap X_0$ is dense in X_0 with respect to the X-norm.*
ii) $(B + F_\sigma)\mathbf{w} \in X_0$ for all $\mathbf{w} \in X_0 \cap D(B)$.

Proof:
First it follows easily that

$$T(t)\mathbf{w} \in X_{0,1} \text{ for all } \mathbf{w} \in X_{0,1} \text{ and } T(t)\mathbf{w} \in X_0 \text{ for all } \mathbf{w} \in X_0. \tag{4.40}$$

Suppose $\mathbf{w} \in X_0$ and define for $h > 0$ $\mathbf{w}_h \stackrel{\text{def}}{=} 1/h \int_0^h T(t)\mathbf{w}dt \in X_0$ by 4.40. Then $\mathbf{w} \in D(B + F_\sigma) = D(B)$ with $(B + F_\sigma)\mathbf{w}_h = 1/h\,(T(h) - 1)\,\mathbf{w}$ and

$$||\mathbf{w}_h - \mathbf{w}||_X \stackrel{\text{h}\to 0}{\longrightarrow} 0.$$

Hence $D(B) \cap X_0$ is dense in X_0. The corresponding assertion for $X_{0,1}$ follows similarely.

In order to prove ii) let $\mathbf{w} \in X_0 \cap D(B)$ and define $\mathbf{u}_h \stackrel{\text{def}}{=} 1/h(T(t) - 1)\mathbf{w}dt \in X_0$ for $h > 0$ by 4.40. Then $||\mathbf{u}_h - (B + F_\sigma)\mathbf{w}||_X \stackrel{\text{h}\to 0}{\longrightarrow} 0$. Since X_0 is closed, it follows $(B + F_\sigma)\mathbf{w} \in X_0$.

□

The aim of the next two lemmata is in particular to show that $||(1 - P)T(t)\mathbf{w}||_X \stackrel{\text{t}\to\infty}{\longrightarrow} 0$ for all $\mathbf{w} \in X_0$.

Lemma 5 *There holds*
i) $\mathcal{N} = (ran\ (B + F_\sigma))^\perp = \ ker\ (B - F_\sigma)$.
ii) $X_{0,1} \subset X_0 \subset \mathcal{N}^\perp$.

Proof:
Since $B^* = -B$, it follows easily from the definition of $\mathcal{N}$ that
$\mathcal{N} \subset \ker\ (B - F_\sigma) \subset (\text{ran}\ (B + F_\sigma))^\perp$.
Now, suppose $\mathbf{w} \in (\text{ran}\ (B + F_\sigma))^\perp = \ker\ (B - F_\sigma)$. Then

$$\int_G \sigma|\underline{\mathbf{w}}_1|^2 dx = \ \text{re}\ \langle \mathbf{w}, F_\sigma \mathbf{w}\rangle_X = \ \text{re}\ \langle \mathbf{w}, B\mathbf{w}\rangle_X = 0,$$

Since $\sigma > 0$ on G, it follows $\underline{\mathbf{w}}_1 = 0$ on G and $F_\sigma\mathbf{w} = 0$ and thus also $B\mathbf{w} = F_\sigma\mathbf{w} = 0$. Hence $\mathbf{w} \in \mathcal{N}$.

In order to prove ii) let $\mathbf{w} \in X_0$. Then

$$\mathbf{u}(t) \stackrel{\text{def}}{=} t^{-1} \int_0^t T(s)\mathbf{w}ds \tag{4.41}$$

obeys

$$||\mathbf{u}(t)||_{L^2(K)} \leq t^{-1} \int_0^t ||T(s)\mathbf{w}||_{L^2(K)}ds$$

$$\leq t^{-1/2}\left(\int_0^t ||T(s)\mathbf{w}||^2_{L^2(K)}ds\right)^{1/2} \overset{t\to\infty}{\longrightarrow} 0 \text{ for all compact sets } K\subset I\!R^3.$$

Hence

$$\mathbf{u}(t) \overset{t\to\infty}{\longrightarrow} 0 \text{ in } X \text{ weakly.} \tag{4.42}$$

For all $\mathbf{a}\in\mathcal{N}\subset\ker B$ one obtains by 2.16

$$\langle T(t)\mathbf{w},\mathbf{a}\rangle_X = \langle \mathbf{w},\mathbf{a}\rangle_X - \int_0^t \langle F_\sigma T(s)\mathbf{w},\mathbf{a}\rangle_X = \langle \mathbf{w},\mathbf{a}\rangle_X$$

and hence

$$\langle \mathbf{u}(t),\mathbf{a}\rangle_X = t^{-1}\int_0^t \langle T(s)\mathbf{w},\mathbf{a}\rangle_X ds = \langle \mathbf{w},\mathbf{a}\rangle_X$$

Now, it follows from 4.42 that

$$\langle \mathbf{w},\mathbf{a}\rangle_X = \langle \mathbf{u}(t),\mathbf{a}\rangle_X \overset{t\to\infty}{\longrightarrow} 0 \tag{4.43}$$

Since $\mathbf{a}\in\mathcal{N}$ is arbitrary, one obtains $\mathbf{w}\in\mathcal{N}^\perp$.

□

Lemma 6 *Let $\mathbf{w}\in\mathcal{N}^\perp$. Then $||(1-P)T(t)\mathbf{w}||_X \overset{t\to\infty}{\longrightarrow} 0$.*

Proof:
It suffices to consider $\mathbf{w}\in$ ran $(B+F_\sigma)$,
since $\mathcal{N}^\perp = \overline{\text{ran }(B+F_\sigma)}$ by Lemma 5 i). Let $\mathbf{w}=(B+F_\sigma)\tilde{\mathbf{w}}$ with $\tilde{\mathbf{w}}\in D(B)$. Then

$$(1-P)T(t)\mathbf{w} = (1-P)\exp\left(t(B+F_\sigma)\right)(B+F_\sigma)\tilde{\mathbf{w}}$$

$$= (1-P)(B+F_\sigma)\exp\left(t(B+F_\sigma)\right)\tilde{\mathbf{w}} = (1-P)F_\sigma T(t)\tilde{\mathbf{w}}.$$

Finally, the assertion follows from Lemma 3 iii).

□

Lemma 7 *Suppose $f\in L^r(\Omega)$ with $r\in[2,\infty)$ and $q\in[1,2]$ with*
$2^{-1}+r^{-1}=q^{-1}$.

Then $||fT(t)\mathbf{w}||_{L^q(\Omega)} \overset{t\to\infty}{\longrightarrow} 0$ for all $\mathbf{w}\in X_{0,1}$.

$t^{-1}\int_0^t ||fT(s)\mathbf{w}||_{L^q(\Omega)}ds \overset{t\to\infty}{\longrightarrow} 0$ *for all* $\mathbf{w}\in X_0$.

and $t^{-1}\int_0^t ||fBT(s)\mathbf{w}||_{L^q(\Omega)}ds \overset{t\to\infty}{\longrightarrow} 0$ *for all* $\mathbf{w}\in X_0\cap D(B)$.

Proof:
Let $R > 0$. Then Hölder's inequality yields

$$||fT(t)\mathbf{w}||_{L^q(\Omega)} \leq ||T(t)\mathbf{w}||_{L^2(\Omega)}||f||_{L^r(\Omega\setminus B_R)}$$

$$+||T(t)\mathbf{w}||_{L^2(\Omega\cap B_R)}||f||_{L^r(\Omega)} \text{ for all } R > 0.$$

Let $\mathbf{w} \in X_{0,1}$. Then the previous estimate yields

$$\overline{\lim_{t\to\infty}}||fT(t)\mathbf{w}||_{L^q(\Omega)} \leq ||\mathbf{w}||_X||f||_{L^r(\Omega\setminus B_R)} \text{ for all } R > 0.$$

By letting $R \longrightarrow \infty$ the first assertion follows. The second one is obtained analogously, whereas the last follows easily from the second together with Lemma 3 iii) and 4 ii), since $BT(t)\mathbf{w} = T(t)(B + F_\sigma)\mathbf{w} - F_\sigma T(t)\mathbf{w}$ for $\mathbf{w} \in X_0 \cap D(B)$.

$\square$

Lemma 8 *There holds*

i) $t^{-1}\int_0^t ||(E-1)T(s)\mathbf{w}||_X ds \overset{t\to\infty}{\longrightarrow} 0$ *for all* $\mathbf{w} \in X_0$.

and ii) $||(E-1)T(t)\mathbf{w}||_X ds \overset{t\to\infty}{\longrightarrow} 0$ *for all* $\mathbf{w} \in X_{0,1}$.

Proof:
Suppose $\mathbf{w} \in D(B)$. Then Lemma 3 ii) and Theorem 2 yield

$$PT(\cdot)\mathbf{w} \in L^\infty((0,\infty), D(B) \cap (\ker\ B)^\perp) \subset L^\infty((0,\infty), L^p(\Omega \setminus B_{R_0})) \tag{4.44}$$

for all $p \in [2, \bar{p}]$ with $\bar{p} \overset{\text{def}}{=} 6/(3-2\bar{s})$, where $\bar{s}$ as in Theorem 2.
Choose $p \in [2, \bar{p}]$ and $r \in [r_0, \infty)$ with $p^{-1} + r^{-1} = 2^{-1}$, where r_0 as in assumption 4.34. Then $E - 1 \in L^{r_0}(I\!R^3) \cap L^\infty(I\!R^3) \subset L^r(I\!R^3)$. Let $R > R_0$. Then Hölder's inequality yields

$$||(E-1)PT(t)\mathbf{w}||_X \leq ||PT(t)\mathbf{w}||_{L^p(\Omega\setminus B_{R_0})}||E-1||_{L^r(\Omega\setminus B_R)} \tag{4.45}$$

$$+||PT(t)\mathbf{w}||_{L^2(\Omega\cap B_R)}||E-1||_{L^\infty(\Omega)}$$

$$\leq ||PT(\cdot)\mathbf{w}||_{L^\infty((0,\infty),L^p(\Omega\setminus B_{R_0}))}||E-1||_{L^r(\Omega\setminus B_R)}$$

$$+||PT(t)\mathbf{w}||_{L^2(\Omega\cap B_R)}||E-1||_{L^\infty(\Omega)} \text{ for all } R > 0.$$

Now, let $\mathbf{w} \in X_{0,1} \cap D(B)$. Then

$$\overline{\lim_{t\to\infty}}||(E-1)PT(t)\mathbf{w}||_X \leq ||PT(\cdot)\mathbf{w}||_{L^\infty((0,\infty),L^p(\Omega\setminus B_{R_0}))}||E-1||_{L^r(\Omega\setminus B_R)}$$

By letting $R \longrightarrow \infty$ one obtains $\lim_{t\to\infty}||(E-1)PT(t)\mathbf{w}||_X = 0$. Since $E \in L^\infty(\Omega)$, the assertion ii) follows together with Lemma 5 ii) and 6.

Part i) follows from similar arguments using

$$\overline{\lim_{t\to\infty}} \left(t^{-1} \int_0^t ||(E-1)PT(s)\mathbf{w}||_X ds \right)$$

$$\leq ||PT(\cdot)\mathbf{w}||_{L^\infty((0,\infty),L^p(\Omega\setminus B_{R_0}))} ||E-1||_{L^r(\Omega\setminus B_R)}$$

for $\mathbf{w} \in X_0 \cap D(B)$.

□

The next theorem says roughly speaking that the asymptotic propagation-speed of the wave-packet $T(t)\mathbf{w}$ does not exceed 1 as $t \to \infty$.

Theorem 4 *Suppose* $\mathbf{w} \in X$ *and* $a > 1$.
Then

$$\int_{\{|x|\geq at\}} |T(t)\mathbf{w}|^2 dx \overset{\mathrm{t}\to\infty}{\longrightarrow} 0.$$

Proof: Since $T(t)\mathbf{w} = \mathbf{w}$ if $\mathbf{w} \in \mathcal{N}$, it suffices to consider $\mathbf{w} \in \mathcal{N}^\perp$. By Lemma 5 i) it suffices to consider $\mathbf{w} \in$ ran $(B + F_\sigma) \cap D(B)$, i.e.

$$\mathbf{w} = (B + F_\sigma)\mathbf{u} \text{ with } \mathbf{u} \in D((B + F_\sigma)^2). \tag{4.46}$$

Let $g \in C^\infty(I\!R)$ with $g(u) = 1$ for $u \geq (1 + a)/2$ and $g(u) = 0$ for $u \leq 1$. For $R > 0$ define

$$\mathcal{E}_R(t) \overset{\text{def}}{=} \left\langle g((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X.$$

Then

$$\begin{aligned}
\mathcal{E}_R'(t) &= 2\left\langle g((t+R)^{-1}|x|)T(t)\mathbf{w}, (B+F_\sigma)T(t)\mathbf{w} \right\rangle_X \qquad (4.47)\\
&-(t+R)^{-2}\left\langle |x|g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X\\
&\leq 2\left\langle g((t+R)^{-1}|x|)ET(t)\mathbf{w}, B_0ET(t)\mathbf{w} \right\rangle_X\\
&-(t+R)^{-2}\left\langle |x|g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X\\
&= (t+R)^{-1}\left\langle S|x|^{-1}g'((t+R)^{-1}|x|)ET(t)\mathbf{w}, ET(t)\mathbf{w} \right\rangle_X\\
&-(t+R)^{-2}\left\langle |x|g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X\\
&= (t+R)^{-1}\left\langle S|x|^{-1}g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X\\
&-(t+R)^{-2}\left\langle |x|g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X + h_R(t)
\end{aligned}$$

$$\leq (t+R)^{-1}\int_{I\!R^3} g'((t+R)^{-1}|x|)(1-(t+R)^{-1}|x|)|T(t)\mathbf{w}|^2dx + h_R(t)$$

$$\leq h_R(t),$$

since $g'((t+R)^{-1}|x|)(t+R)^{-1}|x| \geq g'((t+R)^{-1}|x|)$ by the assumptions on g. Here

$$h_R(t) \stackrel{\text{def}}{=} (t+R)^{-1}\left\langle S|x|^{-1}g'((t+R)^{-1}|x|)ET(t)\mathbf{w}, ET(t)\mathbf{w}\right\rangle_X \tag{4.48}$$

$$-(t+R)^{-1}\left\langle S|x|^{-1}g'((t+R)^{-1}|x|)T(t)\mathbf{w}, T(t)\mathbf{w}\right\rangle_X$$

obeys

$$|h_R(t)| \leq C_1(t+R)^{-1}||(1-E)T(t)\mathbf{w}||_{L^2(\{|x|\geq t+R\})}.$$

Choose $p \in [2,\bar{p}]$ and $r \in [r_0,\infty)$ with $p^{-1}+r^{-1} = 2^{-1}$, where r_0 as in assumption 4.34 and $\bar{p} \stackrel{\text{def}}{=} 6/(3-2\bar{s})$, with $\bar{s}$ as in Theorem 2.
Since $(E-1) \in L^\infty(I\!R^3)$ it follows from 4.34 that

$$(1+|x|)^{\delta_1}(E-1) \in L^r(I\!R^3), \tag{4.49}$$

where $\delta_1 \stackrel{\text{def}}{=} (1+\alpha_0)r_0/r$. Next Lemma 3 ii) and Theorem 2 yield

$$PT(\cdot)\mathbf{w} \in L^\infty((0,\infty), D(B)\cap(\ker\, B)^\perp) \subset L^\infty((0,\infty), L^p(\Omega\setminus B_{R_0})) \tag{4.50}$$

Now, Hölder's inequality yields by 4.46, 4.48 - 4.50

$$|h_{1,R}(t)| \leq C_1(t+R)^{-1}\Big(||(1-E)PT(t)\mathbf{w}||_{L^2(\{|x|\geq t+R\})} \tag{4.51}$$

$$+||(1-E)(1-P)F_\sigma T(t)\mathbf{u}||_X)$$

$$\leq C_2(t+R)^{-1}||(E-1)||_{L^r(\{|x|\geq t+R\})}||PT(t)\mathbf{w}||_{L^p(I\!R^3\setminus B_{R_0})}$$

$$+C_2(t+R)^{-1}||F_\sigma T(t)\mathbf{u}||_X$$

$$\leq C_3(t+R)^{-1-\delta_1}||(1+|x|)^{\delta_1}(E-1)||_{L^r(I\!R^3\setminus B_{R_0}))}||PT(t)\mathbf{w}||_{L^p(I\!R^3\setminus B_{R_0})}$$

$$+C_2(t+R)^{-1}||F_\sigma T(t)\mathbf{u}||_X$$

$$\leq C_4(t+R)^{-1-\delta_1} + C_2(t+R)^{-1}||F_\sigma T(t)\mathbf{u}||_X$$

with some $C_3 \in (0,\infty)$ independent of t, R. Now, it follows from 4.47 and 4.51 that

$$\mathcal{E}_R(t) \leq \mathcal{E}_R(1) + C_4\int_1^t (s+R)^{-1-\delta_1}ds \tag{4.52}$$

$$+C_2\left(\int_1^t (s+R)^{-2}ds\right)^{1/2}\left(\int_1^t ||F_\sigma T(t)\mathbf{u}||_X^2 ds\right)^{1/2}$$

$$\leq \mathcal{E}_R(1) + C_5(1+R)^{-\delta_1}$$

with some $C_5 \in (0,\infty)$ independent of t, R.
Suppose $\delta \geq 0$. Since $g(0) = 0$ it follows $\mathcal{E}_R(1) \overset{\mathrm{R}\to\infty}{\longrightarrow} 0$. By 4.52 there exists $R > 0$ with $\mathcal{E}_R(t) \leq \delta$ and hence

$$\int_{\{|x|\geq(a+1)(t+R)/2\}} |T(t)\mathbf{w}|^2 dx \leq \mathcal{E}_R(t) \leq \delta \text{ for all } t \geq 0.$$

Thus

$$\limsup_{t\to\infty} \int_{\{|x|\geq(a+1)(t+R)/2\}} |T(t)\mathbf{w}|^2 dx = 0.$$

Since $a > 1$ the assertion follows.

$\square$

Recall that $\mathcal{P}_0$ is the orthogonal-projector on $(\ker\,\mathcal{B}_0)^\perp$ and that
$\chi_0 \in C^\infty(I\!R^3)$ with $\chi_0(x) = 1$ on $I\!R^3 \setminus B_{R_0}$ and supp $\chi_0 \subset \Omega$.

Lemma 9 *Suppose* $\mathbf{w} \in D(B)$, *Then*

$$\mathcal{P}_0\chi_0 ET(\cdot)\mathbf{w} \in L^\infty(0,\infty), H^1(I\!R^3)).$$

Proof:
It follows from Lemma 3 ii) that

$$T(\cdot)\mathbf{w} \in L^\infty(0,\infty), D(B)) \text{ and hence } ET(\cdot)\mathbf{w} \in L^\infty(0,\infty), D(B_0)).$$

Since supp $\chi_0 \subset \Omega$, this yields

$$\chi_0 ET(\cdot)\mathbf{w} \in L^\infty(0,\infty), D(\mathcal{B}_0)) \text{ and hence}$$

$$\mathcal{P}_0\chi_0 ET(\cdot)\mathbf{w} \in L^\infty(0,\infty), D(\mathcal{B}_0) \cap (ker\mathcal{B}_0)^\perp) \subset L^\infty(0,\infty), H^1(I\!R^3)).$$

$\square$

Lemma 10 *For all* $\mathbf{w} \in X$ *one has*

$$||\chi_0\mathbf{w} - \mathcal{P}_0\chi_0 E\mathbf{w}||_{L^2(I\!R^3)}$$

$$\leq K_1\left(||(1-E)\mathbf{w}||_X + ||(1-P)\mathbf{w}||_X + ||\mathbf{w}||_{L^2(B_{R_0})}\right)$$

with some constant independent of $\mathbf{w}$.

Proof:
Let $\mathbf{w} \in X$ and define $\mathbf{g} \stackrel{\text{def}}{=} (1-\mathcal{P}_0)\chi_0 E^{-1}\mathbf{w} \in \ker \mathcal{B}_0$, i.e. curl $\underline{\mathbf{g}}_j = 0$ on $I\!R^3$. Hence there exist $\varphi_j \in L^6(I\!R^3)$ with $\nabla\varphi_j \in L^2(I\!R^3))$ such that

$$\underline{\mathbf{g}}_j = \nabla\varphi_j \tag{4.53}$$

Hence

$$||\mathbf{g}||^2_{L^2(I\!R^3)} = \langle \chi_0 E^{-1}\mathbf{w}, \mathbf{g}\rangle_{L^2(I\!R^3)}$$

$$= \int_{I\!R^3} \chi_0 \cdot \left(\underline{E^{-1}\mathbf{w}}_1 \nabla\varphi_1 + \underline{E^{-1}\mathbf{w}}_2 \nabla\varphi_2\right) dx$$

$$= \langle \mathbf{w}, E^{-1}\left(\nabla[\chi_0\varphi_1], \nabla[\chi_0\varphi_2]\right)\rangle_X$$

$$- \int_{I\!R^3} \left(\underline{E^{-1}\mathbf{w}}_1 (\nabla\chi_0)\varphi_1 - \underline{E^{-1}\mathbf{w}}_2 (\nabla\chi_0)\varphi_2\right) dx$$

Since $E^{-1}\left(\nabla[\chi_0\varphi_1], \nabla[\chi_0\varphi_2]\right) \in \ker B$, it follows

$$||(1-\mathcal{P}_0)\chi_0 E^{-1}\mathbf{w}||^2_{L^2(I\!R^3)} = ||\mathbf{g}||^2_{L^2(I\!R^3)} \tag{4.54}$$

$$\leq C_1 ||(1-P)\mathbf{w}||_X \left(||\nabla\varphi_1||_{L^2(I\!R^3)} + ||\varphi_1||_{L^6(I\!R^3)}\right.$$

$$\left. + ||\nabla\varphi_2||_{L^2(I\!R^3)} + ||\varphi_1||_{L^6(I\!R^3)}\right)$$

$$+ C_1 ||\mathbf{w}||_{L^2(B_{R_0})} \left(||\varphi_1||_{L^2(B_{R_0})} + ||\varphi_2||_{L^2(B_{R_0})}\right)$$

$$\leq C_2 \left(||(1-P)\mathbf{w}||_X + ||\mathbf{w}||_{L^2(B_{R_0})}\right) ||\mathbf{g}||_{L^2(I\!R^3)}.$$

Since $E, E^{-1} \in L^\infty(\Omega)$, this completes the proof.

□

Now, it follows immediately from Lemma 5 ii) 6, 8, 10 and Theorem 4 that

Corollary 1 *i) For all* $\mathbf{w} \in X_0$ *one has*

$$t^{-1}\int_0^t ||\chi_0 T(s)\mathbf{w} - \mathcal{P}_0\chi_0 E T(s)\mathbf{w}||_{L^2(I\!R^3)} ds \stackrel{t\to\infty}{\longrightarrow} 0..$$

in particular

$$t^{-1}\int_0^t ||\mathcal{P}_0\chi_0 E T(s)\mathbf{w}||^2_{L^2(I\!R^3)} ds \stackrel{t\to\infty}{\longrightarrow} \mathcal{E}_\infty(\mathbf{w}).$$

and $t^{-1}\int_0^t ||\mathcal{P}_0\chi_0 E T(s)\mathbf{w}||_{L^2(\{|x|\geq as\})} ds \stackrel{t\to\infty}{\longrightarrow} 0$ *for all* $a \in (1,\infty)$.

ii) For all $\mathbf{w} \in X_{0,1}$ *one has*

$$||\chi_0 T(t)\mathbf{w} - \mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(I\!R^3)} ds \stackrel{t\to\infty}{\longrightarrow} 0..$$

in particular

$$||\mathcal{P}_0\chi_0 ET(t)\mathbf{w}||^2_{L^2(I\!R^3)} ds \stackrel{t\to\infty}{\longrightarrow} \mathcal{E}_\infty(\mathbf{w}).$$

and $||\mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(\{|x|\geq at\})} ds \stackrel{t\to\infty}{\longrightarrow} 0$ *for all* $a \in (1,\infty)$.

Lemma 11 *Suppose* $\mathbf{w} \in X_0$ *and let* $g \in C_0^\infty(I\!R)$ *with* $g(u) = 1$ *on a neighbourhood of* $[0,1]$. *Then*

$$t^{-1}\langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)} \stackrel{t\to\infty}{\longrightarrow} \mathcal{E}_\infty(\mathbf{w}),$$

with $\mathcal{E}_\infty(\mathbf{w}) \stackrel{\text{def}}{=} \lim_{t\to\infty} ||T(t)\mathbf{w}||^2_X$.

Proof: Define

$$F(t) \stackrel{\text{def}}{=} \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)}.$$

Then

$$F'(t) = 2\ \text{re}\ \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 E(B+F_\sigma)T(t)\mathbf{w}\rangle_{L^2(I\!R^3)} \tag{4.55}$$

$$-t^{-2}\,\langle S|x|g'(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

$$= h_1(t) + h_2(t) + 2\ \text{re}\ \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{B}_0\mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

$$-t^{-2}\,\langle S|x|g'(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

Here

$$h_1(t) \stackrel{\text{def}}{=} 2\ \text{re}\ \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 EF_\sigma T(t)\mathbf{w}\rangle_{L^2(I\!R^3)}, \tag{4.56}$$

$$h_2(t) \stackrel{\text{def}}{=} 2\ \text{re}\ \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \tag{4.57}$$

$$\left[\mathcal{P}_0\chi_0 E^2 B_0 - \mathcal{B}_0\mathcal{P}_0\chi_0\right] ET(t)\mathbf{w}\Big\rangle_{L^2(I\!R^3)}$$

Suppose $\varphi \in W^{1,\infty}_{loc}(I\!R)$ with $(1+t)\varphi(t) \in L^\infty(0,\infty)$ and $t\varphi'(t) \in L^\infty(0,\infty)$.
For $\mathbf{u} \in (\ \ker\ \mathcal{B}_0)^\perp \cap D(\mathcal{B}_0) \subset H^1(I\!R^3)$ one has $\ \text{div}\ \underline{\mathbf{u}}_j = 0$. Therefore, it follows from the identity
$x\wedge\ \text{curl}\ \mathbf{a} = \nabla(x\mathbf{a}) - \mathbf{a} - (x\nabla)\mathbf{a}$ that

$$\mathcal{P}_0\left(S\varphi(|x|)\mathcal{B}_0\mathbf{u} - \mathcal{B}_0 S\varphi(|x|)\mathbf{u}\right) \tag{4.58}$$

$$= \mathcal{P}_0 \left(\varphi(|x|)\mathbf{u} + |x|\varphi'(|x|)\mathbf{u} - 2|x|^{-1}\varphi'(|x|)\left([x\underline{\mathbf{u}}_1]x, [x\underline{\mathbf{u}}_2]x\right)\right).$$

$$= \mathcal{P}_0 \left(\varphi(|x|)\mathbf{u} + |x|\varphi'(|x|)\mathbf{u} - 2\tilde{S}\varphi'(|x|)\mathbf{u}\right).$$

with $\tilde{S}\mathbf{u} \stackrel{\text{def}}{=} |x|^{-1}\left([x\underline{\mathbf{u}}_1]x, [x\underline{\mathbf{u}}_2]x\right)$.
Hence

$$2 \text{ re } \langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{B}_0\mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

$$= \langle \mathcal{P}_0\left(Sg(|x|/t)\mathcal{B}_0\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \mathcal{B}_0 Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}\right), T(t)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

$$= \left\langle [g(|x|/t) + t^{-1}|x|g'(|x|/t)]\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\right\rangle_{L^2(I\!R^3)}$$

$$-2t^{-1}\left\langle \tilde{S}g'(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\right\rangle_{L^2(I\!R^3)}$$

With 4.55 -4.57 it follows

$$F'(t) = ||\mathcal{P}_0\chi_0 ET(t)\mathbf{w}||^2_{L^2(I\!R^3)} + \sum_{j=1}^{3} h_j(t) \tag{4.59}$$

where

$$h_3(t) \stackrel{\text{def}}{=} \left\langle [g(|x|/t) - 1 + t^{-1}|x|g'(|x|/t)]\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\right\rangle_{L^2(I\!R^3)} \tag{4.60}$$

$$-t^{-1}\left\langle (2\tilde{S} + t^{-1}|x|S)g'(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\right\rangle_{L^2(I\!R^3)}$$

Lemma 2 and Lemma 9 yield

$$|h_1(t)| \leq C_1||(1+|x|)^{-1-\alpha}Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{H^1}$$

$$||(1+|x|)^{1+\alpha_0}\sigma\underline{[T(t)\mathbf{w}]}_1||_{L^{q_0}} \leq C_2||(1+|x|)^{1+\alpha_0}\sigma\underline{[T(t)\mathbf{w}]}_1||_{L^{q_0}}$$

and hence by lemma 7 and Assumption 4.34

$$t^{-1}\int_0^t |h_1(s)|ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.61}$$

Next one obtains by Assumption 4.34 and Hölder's inequality with $q_0 = \left(r_0^{-1} + 2^{-1}\right)^{-1}$ that

$$|h_2(t)| \leq C_3||(1+|x|)^{-1-\alpha}Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{H^1}$$

$$||(1+|x|)^{1+\alpha_0}\left[\chi_0(E^2-1)B_0 + \mathcal{C}_0\right]ET(t)\mathbf{w}||_{L^{q_0}}$$

$$\leq C_4||(1+|x|)^{1+\alpha_0}(E-1)BT(t)\mathbf{w}||_{L^{q_0}} + C_4||T(t)\mathbf{w}||_{L^2(B_{R_0})}$$

and hence by Lemma 7

$$t^{-1}\int_0^t |h_2(s)|ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.62}$$

Since $g'(|x|/t)=0$ and $g(|x|/t)=1$ for $|x|\le at$ with some $a>1$, Corollary 1 i) yields

$$t^{-1}\int_0^t |h_3(s)|ds \le C_5 t^{-1}\int_0^t ||\mathcal{P}_0\chi_0 ET(s)\mathbf{w}||_{L^2(\{|x|\ge as\})}ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.63}$$

Now, it follows from 4.59-4.63 and Corollary 1 that

$$\lim_{t\to\infty} t^{-1}F(t) = \lim_{t\to\infty} t^{-1}\int_1^t F'(s)ds$$

$$= \lim_{t\to\infty} t^{-1}\int_1^t ||\mathcal{P}_0\chi_0 ET(s)\mathbf{w}||^2_{L^2(I\!R^3)}ds = \mathcal{E}_\infty(\mathbf{w}).$$

This completes the proof.

$\square$

Next it is shown that $X_{0,1}=X_0$.

Theorem 5 *Let* $\mathbf{w}\in X_0$. *Then* $\mathbf{w}\in X_{0,1}$, *i.e.*

$$||T(t)\mathbf{w}||_{L^2(\Omega\cap B_R)} \stackrel{t\to\infty}{\longrightarrow} 0 \textit{ for all } R>0.$$

Proof:
Suppose $a>1$. Choose $g\in C_0^\infty(I\!R,[0,\infty))$ with $g(y)=1$ on $[0,b]$ and $g(u)=0$ for all $u>a$ with some $b\in(1,a)$. Then $(t^2-|x|^2/a^2)g(|x|/t)\ge 0$.
Let $f(t)\stackrel{\text{def}}{=}||[(t^2-|x|^2/a^2)g(|x|/t)]^{1/2}T(t)\mathbf{w}||^2_X$.
Then one obtains from 2.15 using the fact that F_σ is nonpositive

$$f'(s) = 2\text{re }\Big\langle (s^2-|x|^2/a^2)g(|x|/s)T(s)\mathbf{w},(B+F_\sigma)T(s)\mathbf{w}\Big\rangle_X \tag{4.64}$$

$$+2s\,\langle g(|x|/s)T(s)\mathbf{w},T(s)\mathbf{w}\rangle_X$$

$$-s^{-2}\Big\langle (s^2-|x|^2/a^2)|x|g'(|x|/s)T(s)\mathbf{w},T(s)\mathbf{w}\Big\rangle_X$$

$$\le 2\text{re }\Big\langle (s^2-|x|^2/a^2)g(|x|/s)ET(s)\mathbf{w},B_0ET(s)\mathbf{w}\Big\rangle_X$$

$$+2s\,\langle g(|x|/s)T(s)\mathbf{w},T(s)\mathbf{w}\rangle_X + C_1 s||T(s)\mathbf{w}||_{L^2(\{|x|\ge bs\})}$$

$$\le -2a^{-2}\,\langle Sg(|x|/s)ET(s)\mathbf{w},ET(s)\mathbf{w}\rangle_X + 2s||T(s)\mathbf{w}||^2_X + C_2 s||T(s)\mathbf{w}||_{L^2(\{|x|\ge bs\})}$$

$$\le 2s||T(s)\mathbf{w}||^2_X - 2a^{-2}\,\langle Sg(|x|/s)\mathcal{P}_0\chi_0 ET(s)\mathbf{w},\mathcal{P}_0\chi_0 ET(s)\mathbf{w}\rangle_{L^2(I\!R^3)}$$

$$+C_3 s \left(||(1-E)T(s)\mathbf{w}||_X + ||(1-\chi_0)T(s)\mathbf{w}||_X\right.$$

$$\left.+||\chi_0 T(s)\mathbf{w} - \mathcal{P}_0\chi_0 ET(s)\mathbf{w}||_{L^2(I\!R^3)} + ||T(s)\mathbf{w}||_{L^2(\{|x|\geq bs\})}\right)$$

Next it follows from Lemma 8 i), Corollary 1 i), Theorem 4 and Lemma 11 that

$$\limsup_{t\to\infty}\left(t^{-1}\int_0^t s^{-1}f'(s)ds\right) \leq 2\limsup_{t\to\infty} t^{-1}\int_0^t \left(||T(s)\mathbf{w}||_X^2\right.$$

$$\left.-a^{-2}s^{-1}\left\langle Sg(|x|/s)\mathcal{P}_0\chi_0 ET(s)\mathbf{w}, \mathcal{P}_0\chi_0 ET(s)\mathbf{w}\right\rangle_{L^2(I\!R^3)}\right) ds$$

$$\leq (1-a^{-2})\mathcal{E}_\infty(\mathbf{w}).$$

Hence

$$\limsup_{t\to\infty} ||[(1-a^{-2}t^{-2}|x|^2)g(|x|/t)]^{1/2}T(t)\mathbf{w}||_X^2 = \limsup_{t\to\infty} t^{-2}f(t)$$

$$\leq (1-a^{-2})\mathcal{E}_\infty(\mathbf{w}).$$

Now, let $K\subset I\!R^3$ be a bounded set. Then

$$\limsup_{t\to\infty} ||T(t)\mathbf{w}||_{L^2(\Omega\cap K)} \tag{4.65}$$

$$= \limsup_{t\to\infty} ||[(1-a^{-2}t^{-2}|x|^2)g(|x|/t)]^{1/2}T(t)\mathbf{w}||_{L^2(\Omega\cap K)}$$

$$\leq \limsup_{t\to\infty} ||[(1-a^{-2}t^{-2}|x|^2)g(|x|/t)]^{1/2}T(t)\mathbf{w}||_X \leq (1-a^{-2})^{1/2}\mathcal{E}_\infty(\mathbf{w})^{1/2}.$$

Letting $a\to 1$ one obtains from 4.65 that

$$\lim_{t\to\infty} ||T(t)\mathbf{w}||_{L^2(\Omega\cap K)} = 0 \text{ for all bounded } K\subset I\!R^3.$$

□

Now the main result of this section 4.35 can be proved.

Theorem 6 *Suppose* $\mathbf{w}\in X_0$, $a<1$ *and* $b>1$.

$$\textit{Then } ||T(t)\mathbf{w}||_{L^2(\Omega\cap B_{at})} + ||T(t)\mathbf{w}||_{L^2(\{|x|>bt\})} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{4.66}$$

$$||t^{-1}S\chi_{\{at\leq|x|\leq bt\}}T(t)\mathbf{w} - T(t)\mathbf{w}||_X \stackrel{t\to\infty}{\longrightarrow} 0$$

$$\textit{and } ||\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \chi_0 T(t)\mathbf{w}||_{L^2(I\!R^3)} \stackrel{t\to\infty}{\longrightarrow} 0.$$

Proof: Suppose $\mathbf{w} \in X_0$. Then $\mathbf{w} \in X_{0,1}$ by the previous Theorem. Hence it follows from Corollary 1 ii) that

$$||\chi_0 \mathcal{P}_0 \chi_0 E T(t)\mathbf{w} - T(t)\mathbf{w}||_X \overset{t\to\infty}{\longrightarrow} 0. \tag{4.67}$$

Next Theorem 4, Lemma 11 and 4.67 yield for all $\beta > 1$

$$\lim_{t\to\infty} t^{-1}\langle S\chi_{\{|x|\le\beta t\}} T(t)\mathbf{w}, T(t)\mathbf{w}\rangle_X \tag{4.68}$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)T(t)\mathbf{w}, T(t)\mathbf{w}\rangle_X$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)\mathcal{P}_0\chi_0 ET(t)\mathbf{w}, \mathcal{P}_0\chi_0 ET(t)\mathbf{w}\rangle_{L^2(I\!R^3)} = \mathcal{E}_\infty(\mathbf{w}).$$

Here $g \in C_0^\infty(I\!R, [0,\infty))$ as in Lemma 11.

Let $\beta > 1$. Then

$$\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx \le \int_{\Omega\cap B_{\beta t}} |T(t)\mathbf{w}|^2 dx - \beta^{-1}t^{-1}\int_{\{at\le|x|\le\beta t\}} |x| \quad |T(t)\mathbf{w}|^2 dx$$

$$\le ||T(t)\mathbf{w}||_X^2 - \int_{\{|x|\ge\beta t\}} |T(t)\mathbf{w}|^2 dx$$

$$-\beta^{-1}t^{-1}\int_{\Omega\cap B_{\beta t}} |x| \quad |T(t)\mathbf{w}|^2 dx + \beta^{-1}a\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx$$

$$\le ||T(t)\mathbf{w}||_X^2 - \int_{\{|x|\ge\beta t\}} |T(t)\mathbf{w}|^2 dx$$

$$-\beta^{-1}t^{-1}\langle S\chi_{\{|x|\le\beta t\}} T(t)\mathbf{w}, T(t)\mathbf{w}\rangle_X + a\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx$$

Hence

$$(1-a)\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx \le ||T(t)\mathbf{w}||_X^2 - \int_{\{|x|\ge\beta t\}} |T(t)\mathbf{w}|^2 dx$$

$$-\beta^{-1}t^{-1}\langle S\chi_{\{|x|\le\beta t\}} T(t)\mathbf{w}, T(t)\mathbf{w}\rangle_X$$

Next it follows from Theorem 4 and 4.68 that

$$(1-a)\overline{\lim_{t\to\infty}}\int_{\Omega\cap B_{at}} |T(t)\mathbf{w}|^2 dx \le (1-\beta^{-1})\mathcal{E}_\infty(\mathbf{w}) \text{ for all } \beta > 1.$$

By letting $\beta \to 1$ this yields

$$||T(t)\mathbf{w}||_{L^2(\Omega\cap B_{at})} \overset{t\to\infty}{\longrightarrow} 0. \tag{4.69}$$

Together with Theorem 4 this completes the proof of the first assertion 4.66.

Suppose $\beta > 1$. Then it follows from Theorem 4 that

$$\overline{\lim_{t\to\infty}} t^{-1} ||S\chi_{\{|x|\le bt\}} T(t)\mathbf{w}||_X \le \overline{\lim_{t\to\infty}} t^{-1} ||S\chi_{\{|x|\le \beta t\}} T(t)\mathbf{w}||_X$$

$$\le \beta \lim_{t\to\infty} ||T(t)\mathbf{w}||_X = \beta \mathcal{E}_\infty(\mathbf{w})^{1/2}$$

Letting $\beta \to 1$ this yields

$$\overline{\lim_{t\to\infty}} t^{-1} ||S\chi_{\{|x|\le bt\}} T(t)\mathbf{w}||_X \le \mathcal{E}_\infty(\mathbf{w})^{1/2}$$

By 4.68 one obtains

$$\overline{\lim_{t\to\infty}} ||t^{-1} S\chi_{\{|x|\le bt\}} T(t)\mathbf{w} - T(t)\mathbf{w}||_X^2$$

$$= \overline{\lim_{t\to\infty}} \Big(t^{-2} ||S\chi_{\{|x|\le bt\}} T(t)\mathbf{w}||_X^2$$

$$-2t^{-1} \left\langle S\chi_{\{|x|\le bt\}} T(t)\mathbf{w}, T(t)\mathbf{w} \right\rangle_X + ||T(t)\mathbf{w}||_X^2 \Big) \le 0.$$

This completes the proof of the second assertion.

□

5 Comparison with the free space problem

In this section the investigation of the asymptotic behaviour is continued towards the following theorem.

Theorem 7 *For all* $\mathbf{w} \in X_0$ *the strong limit*

$$W^+(\mathbf{w}) \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp\left(-t\mathcal{B}_0\right) J^* T(t)\mathbf{w}$$

exists in $L^2(I\!R^3)$ *and* $W^+(\mathbf{w}) \in (ker\ \mathcal{B}_0)^\perp$.

Here $J^* : L^2(\Omega) \to L^2(I\!R^3)$ provides the extension by zero on $I\!R^3 \setminus \Omega$.

It follows from Theorem 6 that for all $a < 1 < b$

$$\lim_{t\to\infty} ||t^{-1} S\chi_{\{at\le|x|\le bt\}} \mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(I\!R^3)} = 0. \tag{5.70}$$

Let g be defined by $g(t,u) \stackrel{\text{def}}{=} c_\alpha t^{-1-\alpha} u^\alpha$ for $u \le (1+\alpha)^{-1}\alpha t$ and $g(t,u) \stackrel{\text{def}}{=} u^{-1}$ for $u \ge (1+\alpha)^{-1}\alpha t$. Here $\alpha \stackrel{\text{def}}{=} \alpha_0/2 > 0$ with α_0 as in assumption 4.34 and $c_\alpha > 0$, such that g is continuous at $u = (1+\alpha)^{-1}\alpha t$.

Since $g(t,t) = t^{-1}$, it follows easily from 5.70, Theorem 4 and 6 that

$$\lim_{t\to\infty} ||Sg(t,|x|)\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(I\!R^3)} = 0,$$

and hence with Theorem 6

$$\lim_{t\to\infty} ||\mathcal{P}_0 Sg(t,|x|)\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - J^*T(t)\mathbf{w}||_{L^2(I\!R^3)} \tag{5.71}$$

$$= \lim_{t\to\infty} ||\mathcal{P}_0 Sg(t,|x|)\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(I\!R^3)}$$

$$\leq \lim_{t\to\infty} ||Sg(t,|x|)\mathcal{P}_0\chi_0 ET(t)\mathbf{w} - \mathcal{P}_0\chi_0 ET(t)\mathbf{w}||_{L^2(I\!R^3)} = 0.$$

Now, it suffices by 5.71 to prove the existence of the limit

$$\lim_{t\to\infty} \exp(-t\mathcal{B}_0)L(t)\chi_0 ET(t)\mathbf{w} \text{ for all } \mathbf{w}\in X_0. \tag{5.72}$$

in order to prove Theorem 7. Here $L(t) \overset{\text{def}}{=} \mathcal{P}_0 Sg(t,|x|)\mathcal{P}_0 \in B(L^2,L^2)$, where g is defined as above.
Define $[\mathbf{f},\mathbf{g}]^{(t)} \overset{\text{def}}{=} \langle \mathcal{B}_0\mathbf{f}, L(t)\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{f}, L(t)\mathcal{B}_0\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{f}, \partial_t L(t)\mathbf{g}\rangle_{L^2(I\!R^3)}$ for $\mathbf{f},\mathbf{g}\in L^2(I\!R^3)$.
Next some properties of L are given.

Lemma 12 *i) There exists a constant $K_2\in(0,\infty)$ with*

$$\langle \mathbf{g}, L(t)\mathbf{h}\rangle_{L^2(I\!R^3)} \leq K_2 t^{-1-\alpha}||\mathcal{P}_0\mathbf{g}||_{H^1(I\!R^3)}||(1+|x|)^{1+\alpha_0}\mathbf{h}||_{L^{q_0}(I\!R^3)}$$

for all $\mathbf{g}\in D(\mathcal{B}_0)$, $\mathbf{h}\in L^2(I\!R^3)$ with $(1+|x|)^{1+\alpha_0}\mathbf{h}\in L^{q_0}(I\!R^3)$.
(Here $q_0\in[6/5,2]$ as in assumption 4.34.)
ii) There exists a constant $K_1\in(0,\infty)$ with

$$||L(t)\mathbf{g}||_{L^2(I\!R^3)} \leq K_1||\mathbf{g}||_{L^2(I\!R^3)} \textit{ for all } \mathbf{g}\in L^2(I\!R^3).$$

iii) $[\mathbf{g},\mathbf{g}]^{(t)}\geq 0$ for all $\mathbf{g}\in L^2(I\!R^3)$.

Proof:
Recall that $\mathcal{P}_0\mathbf{g}\in H^1(I\!R^3)$ for all $\mathbf{g}\in D(\mathcal{B}_0)$. Lemma 2 yields

$$|\langle \mathbf{g}, L(t)\mathbf{h}\rangle_{L^2(I\!R^3)}| = \left|\left\langle (1+|x|)^{1+\alpha}\mathbf{f}, \mathcal{P}_0\mathbf{h}\right\rangle_{L^2(I\!R^3)}\right|$$

$$\leq C_1||\mathbf{f}||_{H^1(I\!R^3)}||(1+|x|)^{1+\alpha_0}\mathbf{h}||_{L^{q_0}(I\!R^3)},$$

where $\mathbf{f} \overset{\text{def}}{=} (1+|x|)^{-1-\alpha}Sg(t,|x|)\mathcal{P}_0\mathbf{g}\in H^1(I\!R^3)$. By the definition of g it follows that

$$|\langle \mathbf{g}, L(t)\mathbf{h}\rangle_{L^2(I\!R^3)}| \leq C_1||x(1+|x|)^{-1-\alpha}g(t,|x|)||_{W^{1,\infty}(I\!R^3)}$$

$$||\mathcal{P}_0\mathbf{g}||_{H^1(I\!R^3)}||(1+|x|)^{1+\alpha_0}\mathbf{h}||_{L^{q_0}(I\!R^3)}$$

$$\leq C_2 t^{-1-\alpha}||\mathcal{P}_0\mathbf{g}||_{H^1(I\!R^3)}||(1+|x|)^{1+\alpha_0}\mathbf{h}||_{L^{q_0}(I\!R^3)}$$

whence i).

Since $||xg(t,|x|)||_{L^\infty(I\!R^3)} \leq 1$ one obtains ii) immediately.
It follows from 4.58 that

$$\langle \mathcal{B}_0\mathbf{g}, L(t)\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{g}, L(t)\mathcal{B}_0\mathbf{g}\rangle_{L^2(I\!R^3)}$$

$$= \langle \mathbf{g}, \mathcal{P}_0\left(Sg(t,|x|)\mathcal{B}_0\mathcal{P}_0\mathbf{g} - \mathcal{B}_0 Sg(t,|x|)\mathcal{P}_0\mathbf{g}\right)\rangle_{L^2(I\!R^3)}$$

$$= \left\langle \mathbf{g}, \mathcal{P}_0\left(\left(g(t,|x|) + |x|g_u(t,|x|)\right)\mathcal{P}_0\mathbf{g} - 2\tilde{S}g_u(t,|x|)\mathcal{P}_0\mathbf{g}\right)\right\rangle_{L^2(I\!R^3)}$$

with the positive semidefinite operator $\tilde{S}\mathbf{u} \stackrel{\text{def}}{=} |x|^{-1}\left([x\underline{\mathbf{u}}_1]x, [x\underline{\mathbf{u}}_2]x\right)$.
For $|x| \geq (1+\alpha)^{-1}\alpha t$ one has $g(t,|x|) + |x|g_u(t,|x|) \geq 0$ and $g_t(t,|x|) = 0$, whereas $g(t,|x|) - |xg_u(t,|x|)| - |xg_t(t,|x|) \geq 0$ for $|x| \leq (1+\alpha)^{-1}\alpha t$.
(Without loss of generality it can be assumed that $\alpha \leq 1/2$ in 4.34.) Hence

$$[\mathbf{g},\mathbf{g}]^{(t)} = \langle \mathcal{B}_0\mathbf{g}, L(t)\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{g}, L(t)\mathcal{B}_0\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathcal{B}_0\mathbf{g}, \partial_t L(t)\mathbf{g}\rangle_{L^2(I\!R^3)}$$

$$= \left\langle \mathcal{P}_0\mathbf{g}, \left(\left(g(t,|x|) + |x|g_u(t,|x|)\right)\mathcal{P}_0\mathbf{g} - 2\tilde{S}g_u(t,|x|)\mathcal{P}_0\mathbf{g} + Sg_t(t,|x|)\mathcal{P}_0\mathbf{g}\right)\right\rangle_{L^2(I\!R^3)}$$

$$\geq -2\left\langle \mathcal{P}_0\mathbf{g}, \chi_{\{|x|\geq(1+\alpha)^{-1}\alpha t\}}\tilde{S}g_u(t,|x|)\mathcal{P}_0\mathbf{g}\right\rangle_{L^2(I\!R^3)}$$

$$+\Big\langle \mathcal{P}_0\mathbf{g}, \chi_{\{|x|\leq(1+\alpha)^{-1}\alpha t\}}\left(\left(g(t,|x|) + |x|g_u(t,|x|)\right)\mathcal{P}_0\mathbf{g}\right.$$

$$\left.-2\tilde{S}g_u(t,|x|)\mathcal{P}_0\mathbf{g} + Sg_t(t,|x|)\mathcal{P}_0\mathbf{g}\right)\Big\rangle_{L^2(I\!R^3)}$$

$$\geq \left\langle \mathcal{P}_0\mathbf{g}, \chi_{\{|x|\leq(1+\alpha)^{-1}\alpha t\}}\left(\left(g(t,|x|) - |xg_u(t,|x|)| - |xg_t(t,|x|)|\right)\mathcal{P}_0\mathbf{g}\right)\right\rangle_{L^2(I\!R^3)} \geq 0.$$

□

Next a general principle regarding to the existence of the limit 5.72 is given.

Theorem 8 *Suppose* $\mathbf{u} \in L^\infty((0,\infty), D(B)) \cap C([0,\infty), X)$ *be a weak solution of* $\partial_t\mathbf{u} = B\mathbf{u} + \mathbf{f}$ *where* $\mathbf{f} \in L^1_{loc}([0,\infty), X)$ *obeys* $(1+|x|))^{1+\alpha_0}\mathbf{f} \in L^1((0,\infty), L^{q_0}(\Omega)) + L^\infty((0,\infty), L^{q_0}(\Omega))$.

Then the strong limit

$$\lim_{t\to\infty} \exp\left(-t\mathcal{B}_0\right)L(t)\chi_0 E\mathbf{u}(t) \tag{5.73}$$

exists with respect to the $L^2(I\!R^3)$*-topology.*

Proof:
First it is shown that

$$\int_1^\infty [\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)}\, dt < \infty. \tag{5.74}$$

Let $F(t) \stackrel{\text{def}}{=} \langle \chi_0 E\mathbf{u}(t), L(t)\chi_0 E\mathbf{u}(t)\rangle_{L^2(I\!R^3)}$. Then

$$F'(t) = 2\mathrm{re}\ \langle \chi_0 E\mathbf{u}(t), L(t)\chi_0 E(B\mathbf{u}(t) + \mathbf{f}(t))\rangle_{L^2(I\!R^3)} \tag{5.75}$$

$$+ \langle \chi_0 E\mathbf{u}, \partial_t L(t)\chi_0 E\mathbf{u}(t)\rangle_{L^2(I\!R^3)}$$

$$= [\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)} + \sum_{k=1}^{2} g_k(t),$$

where

$$g_1(t) \stackrel{\text{def}}{=} 2\mathrm{re}\ \langle \chi_0 E\mathbf{u}(t), L(t)\chi_0 E\mathbf{f}(t)\rangle_{L^2(I\!R^3)} \tag{5.76}$$

$$g_2(t) \stackrel{\text{def}}{=} 2\mathrm{re}\ \left\langle \chi_0 E\mathbf{u}(t), L(t)\left(\chi_0 E^2 B_0 E\mathbf{u}(t) - \mathcal{B}_0\chi_0 E\mathbf{u}(t)\right)\right\rangle_{L^2(I\!R^3)} \tag{5.77}$$

Lemma 12 i) and 9 yield

$$|g_1(t)| \leq 2K_1 t^{-1-\alpha}||\mathcal{P}_0\chi_0 E\mathbf{u}(t)||_{H^1(I\!R^3)}||(1+|x|)^{1+\alpha_0}\chi_0 E\mathbf{f}(t)||_{L^{q_0}(I\!R^3)} \tag{5.78}$$

$$\leq C_1 t^{-1-\alpha}(1 + f(t))$$

with some $f \in L^1(0,\infty)$. Similarly one obtains from assumption 4.34 and Hölder's inequality with $1/q_0 = 1/r_0 + 1/2$ the estimate

$$|g_2(t)| \leq 2K_1 t^{-1-\alpha}||\mathcal{P}_0\chi_0 E\mathbf{u}(t)||_{H^1(I\!R^3)} \tag{5.79}$$

$$||(1+|x|)^{1+\alpha_0}\left(\chi_0 E^2 B_0 E\mathbf{u}(t) - \mathcal{B}_0\chi_0 E\mathbf{u}(t)\right)||_{L^{q_0}(I\!R^3)}$$

$$\leq C_2 t^{-1-\alpha}\Big(||(1+|x|)^{1+\alpha_0}\chi_0(E - E^{-1})B\mathbf{u}(t)||_{L^{q_0}(I\!R^3)}$$

$$+||(1+|x|)^{1+\alpha_0}\mathcal{C}_0 E\mathbf{u}(t)||_{L^{q_0}(I\!R^3)}\Big) \leq C_3 t^{-1-\alpha}$$

Now, it follows from 5.75-5.79 that

$$\int_1^\tau [\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)}\, dt \leq \int_1^\tau (C_1 + C_3 + C_1 f(t))t^{-1-\alpha}dt + |F(\tau) - F(1)|$$

Together with Lemma 12 ii) this completes the proof of 5.74.

Next it is shown that there exist a constant $K_2 \in (0,\infty)$, such that

$$\int_1^{\infty} [\exp(t\mathcal{B}_0)\mathbf{g}, \exp(t\mathcal{B}_0)\mathbf{g}]^{(t)}\, dt \le K_2 ||\mathbf{g}||^2_{L^2(I\!R^3)} \text{ for all } \mathbf{g} \in L^2(I\!R^3). \tag{5.80}$$

Suppose $\mathbf{g} \in D(\mathcal{B}_0)$. Then

$$\frac{d}{dt}\langle \exp(t\mathcal{B}_0)\mathbf{g}, L(t)\exp(t\mathcal{B}_0)\mathbf{g}\rangle_{L^2(I\!R^3)}$$

$$= 2\mathrm{re}\ \langle \exp(t\mathcal{B}_0)\mathbf{g}, L(t)\mathcal{B}_0\exp(t\mathcal{B}_0)\mathbf{g}\rangle_{L^2(I\!R^3)}$$

$$+ \langle \exp(t\mathcal{B}_0)\mathbf{g}, \partial_t L(t)\exp(t\mathcal{B}_0)\mathbf{g}\rangle_{L^2(I\!R^3)} = [\exp(t\mathcal{B}_0)\mathbf{g}, \exp(t\mathcal{B}_0)\mathbf{g}]^{(t)}.$$

Finally, 5.80 follows from Lemma 12 ii). Now the existence of the limit 5.73 can be proved. Let $\mathbf{F}(t) \stackrel{\text{def}}{=} \exp(-t\mathcal{B}_0)L(t)\chi_0 E\mathbf{u}(t)$. Then it follows for all $\mathbf{g} \in L^2(I\!R^3)$

$$\frac{d}{dt}\left\langle \mathbf{g}, (1-\mathcal{B}_0)^{-1}\mathbf{F}(t)\right\rangle_{L^2(I\!R^3)} \tag{5.81}$$

$$= \left\langle (1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, L(t)\chi_0 E(B\mathbf{u}(t)+\mathbf{f}(t))\right\rangle_{L^2(I\!R^3)}$$

$$+ \left\langle \mathcal{B}_0(1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, L(t)\chi_0 E\mathbf{u}(t)\right\rangle_{L^2(I\!R^3)}$$

$$+ \left\langle (1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, \partial_t L(t)\chi_0 E\mathbf{u}(t)\right\rangle_{L^2(I\!R^3)}$$

$$= \left[(1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, \chi_0 E\mathbf{u}(t)\right]^{(t)} + \sum_{k=1}^{2} h_k(t),$$

where

$$h_1(t) \stackrel{\text{def}}{=} \left\langle (1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, L(t)\chi_0 E\mathbf{f}(t)\right\rangle_{L^2(I\!R^3)} \tag{5.82}$$

$$h_2(t) \stackrel{\text{def}}{=} \Big\langle (1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, \tag{5.83}$$

$$L(t)\left(\chi_0 E^2 B_0 E\mathbf{u}(t) - \mathcal{B}_0\chi_0 E\mathbf{u}(t)\right)\Big\rangle_{L^2(I\!R^3)}.$$

Lemma 12 i) and 9 yield

$$|h_1(t)| \le 2K_1 t^{-1-\alpha}||\mathcal{P}_0(1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}||_{H^1(I\!R^3)} \tag{5.84}$$

$$||(1+|x|)^{1+\alpha_0}\chi_0 E\mathbf{f}(t)||_{L^{q_0}(I\!R^3)} \le C_4 t^{-1-\alpha}||\mathbf{g}||_{L^2(I\!R^3)}(1+f(t))$$

with some $f \in L^1(0,\infty)$. Similarely

$$|h_2(t)| \leq 2K_1 t^{-1-\alpha} ||\mathcal{P}_0(1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}||_{H^1(I\!R^3)} \tag{5.85}$$

$$||(1+|x|)^{1+\alpha_0}\left(\chi_0 E^2 B_0 E\mathbf{u}(t) - \mathcal{B}_0\chi_0 E\mathbf{u}(t)\right)||_{L^{q_0}(I\!R^3)}$$

$$\leq C_5 t^{-1-\alpha}||\mathbf{g}||_{L^2(I\!R^3)}\left(||(1+|x|)^{1+\alpha_0}\chi_0(E-E^{-1})B\mathbf{u}(t)||_{L^{q_0}(I\!R^3)}\right.$$

$$\left.+||(1+|x|)^{1+\alpha_0}\mathcal{C}_0 E\mathbf{u}(t)||_{L^{q_0}(I\!R^3)}\right) \leq C_6 t^{-1-\alpha}||\mathbf{g}||_{L^2(I\!R^3)}$$

Now, it follows from 5.74-5.80 , 5.81-5.85 and Lemma 12 ii)

$$|\langle \mathbf{g}, (1-\mathcal{B}_0)^{-1}(\mathbf{F}(t_2)-\mathbf{F}(t_1))\rangle_{L^2(I\!R^3)}| \tag{5.86}$$

$$\leq \left(\int_{t_1}^{t_2}\left[(1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}, (1+\mathcal{B}_0)^{-1}\exp(t\mathcal{B}_0)\mathbf{g}\right]^{(t)} dt\right)^{1/2}$$

$$\left(\int_{t_1}^{t_2}[\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)} dt\right)^{1/2}$$

$$+C_7||\mathbf{g}||_{L^2(I\!R^3)}\int_{t_1}^{t_2}(1+f(t))t^{-1-\alpha}dt$$

$$\leq C_8||\mathbf{g}||_{L^2(I\!R^3)}\left(\left(\int_{t_1}^{t_2}[\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)} dt\right)^{1/2} + \int_{t_1}^{t_2}(1+f(t))t^{-1-\alpha}dt\right)$$

with some constant C_8 independent of $t_1, t_2, \mathbf{g}$. Hence,

$$||(1-\mathcal{B}_0)^{-1}(\mathbf{F}(t_2)-\mathbf{F}(t_1))||_{L^2(I\!R^3)} \tag{5.87}$$

$$\leq C_8\left(\left(\int_{t_1}^{t_2}[\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)} dt\right)^{1/2} + \int_{t_1}^{t_2}(1+f(t))t^{-1-\alpha}dt\right)$$

Since $F \in L^\infty((0,\infty), D(\mathcal{B}_0))$ by the assumptions on g and $\mathbf{u}$, 5.87 yields

$$||\mathbf{F}(t_2)-\mathbf{F}(t_1)||^2_{L^2(I\!R^3)} \tag{5.88}$$

$$= \left\langle (1+\mathcal{B}_0)(\mathbf{F}(t_2)-\mathbf{F}(t_1)), (1-\mathcal{B}_0)^{-1}(\mathbf{F}(t_2)-\mathbf{F}(t_1))\right\rangle_{L^2(I\!R^3)}$$

$$\leq 2||(1+\mathcal{B}_0)\mathbf{F}(\cdot)||_{L^\infty((0,\infty),L^2(I\!R^3))}||(1-\mathcal{B}_0)^{-1}(\mathbf{F}(t_2)-\mathbf{F}(t_1))||_{L^2(I\!R^3)}$$

$$\leq C_9\left(\left(\int_{t_1}^{t_2}[\chi_0 E\mathbf{u}(t), \chi_0 E\mathbf{u}(t)]^{(t)} dt\right)^{1/2} + \int_{t_1}^{t_2}(1+f(t))t^{-1-\alpha}dt\right)$$

Finally, the existence of the strong limit $\lim_{t\to\infty} \mathbf{F}(t)$ follows from 5.88 and 5.74 .

$\square$

Proof of Theorem 7:
Suppose $\mathbf{w} \in X_0 \cap D(B)$ and let $\mathbf{u}(t) \stackrel{\text{def}}{=} T(t)\mathbf{w}$ and $\mathbf{f}(t) \stackrel{\text{def}}{=} F_\sigma T(t)\mathbf{w}$. With 4.34 and Lemma 3 one has $(1+|x|))^{1+\alpha_0}\mathbf{f} \in L^\infty((0,\infty), L^{q_0}(\Omega))$. Hence $\mathbf{u}$ satisfies the conditions of the previous theorem, which yields together with 5.71 the existence of the limit

$$\mathbf{U} \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp\left(-t\mathcal{B}_0\right) J^* T(t)\mathbf{w} \text{ in } L^2(I\!\!R^3). \tag{5.89}$$

Since ran $\mathcal{P}_0 \subset$ ran $\mathcal{B}_0$, it follows from 5.71 and 5.89 that
$\mathbf{U} \in \overline{\text{ ran } \mathcal{B}_0} = (\ker \mathcal{B}_0)^\perp$, i.e. div $(\underline{\mathbf{U}}_j) = 0$ on $I\!\!R^3$.

$\square$

6 Sufficient conditions for local decay

First the following assumption will be imposed. There is a subset $G \subset \Omega$ with nonempty interior, such that $\varepsilon(x) = \mu(x) = 1$ and $\sigma(x) = 0$ on $\Omega \setminus G$, whereas $\sigma(x) > 0$ for all $x \in G$. Recall that $\mathcal{N}$ is the set of all $\mathbf{w} \in \ker B$, such that $\underline{\mathbf{w}}_1(x) = 0$ for all $x \in G$. In [14] the following has been proved.

Theorem 9 *Suppose* $\mathbf{w} \in X$.
Then $T(t)\mathbf{w} \stackrel{t\to\infty}{\longrightarrow} 0$ *in* X *weakly, if and only if* $\mathbf{w} \in \mathcal{N}^\perp$.

The proof of 9 in [14] is based on a suitable modification of the approach in [5] for the case that the operator B does not necessarily have purely discrete spectrum. The basic idea is to show that for each $f \in C_0^\infty(I\!\!R \setminus \{0\})$ and $\mathbf{g} \in \omega_0(\mathbf{E}_0, \mathbf{F}_0)$ the function $f(iB)\mathbf{g}$ is real-analytic and vanishes on G, where $\omega_0(\mathbf{E}_0, \mathbf{F}_0)$ denotes the ω-limit-set with respect to the weak topology of the orbit belonging to the initial-state $(\mathbf{E}_0, \mathbf{F}_0)$. This implies $f(iB)\mathbf{g} = 0$ for all $f \in C_0^\infty(I\!\!R \setminus \{0\})$ and hence $\mathbf{g} \in \ker B$. (Here the operator $f(iB)$ can be defined by the spectral-theorem, since iB is self-adjoint in $L^2(\Omega, \mathbb{C}^{M+N})$.)

Theorem 10 *Let* $\mathbf{w} \in \mathcal{N}^\perp$.
Then $||T(t)\mathbf{w}||_{L^2(\Omega\cap B_R)} \stackrel{t\to\infty}{\longrightarrow} 0$ *for all* $R > 0$, *in particular* $\mathbf{w} \in X_0$.

Proof:
By Lemma 6 it suffices to show

$$||PT(t)\mathbf{w}||_{L^2(\Omega\cap B_R)} \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } \mathbf{w} \in \mathcal{N}^\perp \cap D(B). \tag{6.90}$$

For $\mathbf{u} \in (\ker B)^\perp$ one has

$$\text{div } (\varepsilon^{1/2}\underline{\mathbf{u}}_1) = 0, \quad \text{div } (\mu^{1/2}\underline{\mathbf{u}}_2) = 0 \tag{6.91}$$

with $\vec{n}\underline{\mathbf{u}}_1 = 0$ on Γ_2 and $\vec{n}\underline{\mathbf{u}}_2 = 0$ on Γ_1

in the sense that

$$\int_\Omega \left(\varepsilon^{1/2}\underline{\mathbf{u}}_1 \nabla\varphi + \mu^{1/2}\underline{\mathbf{u}}_2 \nabla\psi\right) dx = 0$$

for all $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_1})$ and $\psi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2})$.
This follows from the fact that $(\varepsilon^{1/2}\nabla\varphi, \mu^{1/2}\nabla\psi) \in$ ker B
for all $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_1})$ and $\psi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2})$.

Suppose $\mathbf{u} \in (\ker B)^\perp \cap D(B)$. Then $\underline{\mathbf{u}}_1 \in W_E$, whereas $\underline{\mathbf{u}}_2 \in W_H$. Therefore 6.91 and the compactness-theorem in [9], a generalization of the result in [25], implie that

$$(\ker B)^\perp \cap D(B) \text{ is compactly imbedded in } L^2(\Omega \cap B_R) \text{ for all } R > 0. \tag{6.92}$$

Now, 6.90 follows from Lemma 3, Theorem 9 and 6.92.

□

By Theorem 7 the following result can be summarized.

Corollary 2 *For all* $\mathbf{w} \in \mathcal{N}^\perp$ *the strong limit*

$$W^+(\mathbf{w}) \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-t\mathcal{B}_0) J^* T(t)\mathbf{w}$$

exists in $L^2(I\!R^3)$ *and* $W^+(\mathbf{w}) \in (\ker \mathcal{B}_0)^\perp$.

Remark 1 *The physical meaning of the condition* $\varepsilon(x) = \mu(x) = 1$ *and* $\sigma(x) = 0$ *on* $\Omega_0 \stackrel{\text{def}}{=} \Omega\setminus G$ *is that* Ω_0 *represents the vacuum-region without electrical conductivity. From the mathematical point of view this condition is only required for the weak decay principle Theorem 9.*
As mentioned above the basic idea of the proof of Theorem 9 in [14] is to show that each $\mathbf{g} \in \omega_0(\mathbf{w})$ *satisfies*

$$\underline{(\exp(tB)\mathbf{g})}_1 = 0 \text{ on } G \text{ for all } t \in I\!R. \tag{6.93}$$

and hence

$$f(iB)\mathbf{g} = 0 \text{ on } G \text{ for all } f \in C_0^\infty(I\!R \setminus \{0\}). \tag{6.94}$$

Here $\omega_0(\mathbf{w})$ *denotes the* ω*-limit-set of the solution* $T(\cdot)\mathbf{w}$ *with respect to the weak topology of* X.
Now it follows from 6.94 and the assumption $\varepsilon(x) = \mu(x) = 1$ *on* $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus G$ *that*

$$\tilde{f}(iB)\mathbf{g} = iBf(iB)\mathbf{g} = i\left(\ \mathit{curl}\ \underline{(f(iB)\mathbf{g})}_2, -\ \mathit{curl}\ \underline{(f(iB)\mathbf{g})}_1\right) \tag{6.95}$$

$$\text{and }\ \mathit{div}\ \underline{(f(iB)\mathbf{g})}_1 = \ \mathit{div}\ \underline{(f(iB)\mathbf{g})}_2 = 0 \text{ on } \Omega$$

for all $f \in C_0^\infty(I\!R \setminus \{0\})$*, where* $\tilde{f}(\lambda) = \lambda f(\lambda)$.

In particular $h(iB)\mathbf{g} = -\Delta f(iB)\mathbf{g}$ *on* Ω

where $h(\lambda) = \lambda^2 f(\lambda)$. *Using 6.95 it is shown that* $f(iB)\mathbf{g}$ *is real analytic on* Ω*, which implies* $f(iB)\mathbf{g} = 0$ *by 6.94. The abovementioned analycity-property is not avialable if* ε *and* μ *are not assumed to be smooth on* $\Omega_0 = \Omega \setminus G$.

7 Application to the wave-operators

In this section the undamped case $\sigma = 0$ is considered. First the proof of the existence of the wave-operators

$$\Omega^+ \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-tB) J \exp(t\mathcal{B}_0)\mathcal{P}_0 \tag{7.96}$$

is sketched. Let $U \subset L^2(I\!R^3)$ be the space of all $\mathbf{u} \in \mathcal{S}(I\!R^3)$ with supp $\hat{\mathbf{u}} \subset I\!R^3 \setminus \{0\}$. Then 3.21 yields

$$\mathcal{P}_0(\mathbf{u}) \in U \subset \mathcal{S}(I\!R^3) \text{ for all } \mathbf{u} \in U. \tag{7.97}$$

Choose $\alpha \in (0,1)$ and suppose $\mathbf{U} \in U$. Then it follows from 7.97 and the explicit representaion of $\exp(t\mathcal{B}_0)\mathbf{u} \in U$, in particular Huygen's principle, see [16] that

$$\begin{aligned} &||\exp(t\mathcal{B}_0)\mathcal{B}_0\mathbf{u}||_{L^2(B_{\alpha t})} + ||\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{u}||_{L^2(B_{\alpha t})} \\ &\leq C_1(1+t^2)^{-1} \text{ for all } t \geq 1. \end{aligned} \tag{7.98}$$

This implies by a similar argument as in Lemma 8 that

$$||(E^{-1}\chi_0 - J)\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_X \stackrel{\mathrm{t}\to\infty}{\longrightarrow} 0$$

Hence it suffices to prove the existence of the limit

$$\lim_{t\to\infty} \exp(-tB)E^{-1}\chi_0 \exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U} \tag{7.99}$$

in the X-topology. This will be achieved by 'Cook's method', which means that

$$\partial_t \exp(-tB)E^{-1}\chi_0 \exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U} \in L^1((1,\infty), X). \tag{7.100}$$

With 4.34, 7.98, Sobolev's embedding theorem and Hölder's inequality one obtains

$$\begin{aligned} &||\partial_t \exp(-tB)E^{-1}\chi_0 \exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_X \\ &= ||(E^{-1}\chi_0\mathcal{B}_0 - EB_0\chi_0)\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_X \\ &= ||(E - E^{-1})\chi_0 \exp(t\mathcal{B}_0)\mathcal{B}_0\mathbf{U}||_X + ||E\mathcal{C}_0 \exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_X \\ &\leq C_2||E-1||_{L^{r_0}(I\!R^3\setminus B_{\alpha t})}||\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_{H^1(I\!R^3)} \\ &\quad + C_2||E-1||_{L^\infty(\Omega)}||\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_{L^2(B_{\alpha t})} \\ &\leq C_3 t^{-1-\alpha_0} + C_3||\exp(t\mathcal{B}_0)\mathcal{P}_0\mathbf{U}||_{L^2(B_{\alpha t})} \leq C_4 t^{-1-\alpha_0} \end{aligned}$$

This proves 7.100 and thus the existence of the limit 7.99. Next ran Ω^+ is examined.

Corollary 3 *For all* $\mathbf{w} \in X_{cont}(iB)$ *the strong limit*

$$W^+(\mathbf{w}) \stackrel{\text{def}}{=} \lim_{t \to \infty} \exp(-t\mathcal{B}_0) J^* T(t)\mathbf{w}$$

exists in $L^2(I\!R^3)$ *and* $W^+(\mathbf{w}) \in (ker\ \mathcal{B}_0)^\perp$.

Proof:
By Theorem 7 it suffices to to show that $X_{cont}(iB) \subset X_0$. But this follows from RAGE-theorem, see [22] together with the local compactness-property 6.92.

□

The result of this section is

Theorem 11 *ran* $\Omega^+ = X_{ac}(iB) = X_{cont}(iB)$.

Proof:
Since $\mathcal{X}_{ac}(i\mathcal{B}_0) = \mathcal{X}_{cont}(i\mathcal{B}_0) = (\ker \mathcal{B}_0)^\perp$, it follows ran $\Omega^+ \subset X_{ac}(B)$. It remains to show

$$X_{cont}(iB) \subset \text{ ran } \Omega^+. \tag{7.101}$$

Suppose $\mathbf{w} \in X_{cont}(iB)$. By Corollary 3 $\mathbf{u} \stackrel{\text{def}}{=} W^+(\mathbf{w}) \in (\ker \mathcal{B}_0)^\perp$ satisfies

$$||\exp(-tB) J \exp(t\mathcal{B}_0)\mathbf{u} - \mathbf{w}||_X = ||J \exp(t\mathcal{B}_0)\mathbf{u} - \exp(tB)\mathbf{w}||_X$$

$$\leq ||\exp(t\mathcal{B}_0)\mathbf{u} - J^* \exp(tB)\mathbf{w}||_{L^2(I\!R^3)}$$

$$= ||\mathbf{u} - \exp(-t\mathcal{B}_0) J^* \exp(tB)\mathbf{w}||_{L^2(I\!R^3)} \stackrel{t \to \infty}{\longrightarrow} 0.$$

Hence $\mathbf{w} = \Omega^+\mathbf{u} \in$ ran Ω^+, whence 7.101.

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